THE FIRST JOSIAH WILLARD GIBBS LECTURE

The first Josiah Willard Gibbs Lecture was delivered under the auspices of this Society on February 29, 1924, by Professor M. I. Pupin, of Columbia University, in the auditorium of the Engineering Societies' Building, New York City. A large and distinguished audience was present, including, besides members of the Society, many physicists, chemists, and engineers who had been invited to attend.

In introducing the speaker, President Veblen spoke as follows:

"In instituting the Willard Gibbs Lectures, the American Mathematical Society has recognized the dual character of mathematics. On the one hand, mathematics is one of the essential emanations of the human spirit,—a thing to be valued in and for itself, like art or poetry. Gibbs made notable contributions to this side of mathematics in his work on vector analysis and multiple algebras.

"On the other hand, mathematics is the handmaiden and helper of the other sciences, both in their most abstract generalizations and in their most concrete applications to industry. In this field Gibbs may be justly described as transcendental,—even if we think only of his work in thermodynamics. For his paper *On the equilibrium of heterogeneous substances* is one of the foundation stones of physical chemistry.

"It is hoped that the Willard Gibbs Lectures will remind the mathematicians of something that we fear they sometimes forget,—the existence of an outside world. It is equally hoped that they will remind the outside world that mathematics is a going concern,—not a pedantic exercise for the torment of school boys, but a living organism growing larger and stronger each year. Also it is intended that these should be lectures which we can all understand. We hope not to have an experience like one I heard of

last summer. A friend of mine, an astronomer, was describing to me his attempt to read a book by Einstein. At a certain point Einstein says 'Until you have grasped this point, dear reader, read no further.' 'At this point,' said my friend, 'I closed the book and have never opened it again.'

"In asking Professor Pupin to give the first Willard Gibbs Lecture we have made sure that we shall not be tempted to close the book before we finish it. For we all know him not only as an eminent physicist and electrical engineer, but also as a rich and vivid personality and a teacher who has inspired more than one generation of students. He is well known to you as a man who has made brilliant applications of mathematics to electrical problems. But on this occasion we recall with special pleasure his connection with the American Mathematical Society. He is one of its charter members. He contributed a very famous paper (*Wave-propagations over non-uniform electrical conductors*) to the first volume of its TRANSACTIONS, and since the recent incorporation of the Society he has been one of its trustees. May I introduce our fellow mathematician, Professor Pupin."

The very illuminating lecture that followed marked the inauguration of what is confidently expected to be a notable series contributing to an appreciation on the part of the educated public of the role that mathematics plays in modern thought. Under the title *Coordination*, the lecturer divided all physical phenomena into two groups: first, coordinated phenomena, illustrated by planetary motions, and second, non-coordinated phenomena, illustrated by molecular motions. Newton, Faraday, and Maxwell laid the foundations for the dynamics of coordinated phenomena, and prepared the way for the study of non-coordinated phenomena. Sadi Carnot, with his second law of thermodynamics, led the way into the study of the dynamics of non-coordination. The discovery of the fundamental laws of radiation by Kirchhoff, and the development of the kinetic theory of gases by Maxwell, Boltzmann, and Clausius, gradually disclosed the physical fact that all radiation from

hot bodies is a non-coordinated process, and that by far the greatest quantities of energy in the universe are in a non-coordinated form. The lecturer showed that the work of Gibbs in thermodynamics, in particular his generalization of Carnot's law, entitles him to the position of one of the founders of the science of non-coordinated motions.

R. G. D. RICHARDSON,
Secretary.

THE MARCH MEETING OF THE SOCIETY

The two hundred thirty-fourth regular meeting of the Society was held at Columbia University on Saturday, March 1, and was preceded, on February 29, by the first Josiah Willard Gibbs Lecture (see this BULLETIN, p. 289). The attendance at this meeting included the following forty-seven members of the Society:

Alexander, Babb, Barnum, Bernstein, W. M. Bond, E. W. Brown, R. W. Burgess, Coolidge, Cowley, Eisenhart, Fields, Fite, Gafafer, Gehman, Gilman, Glenn, Grove, Guggenbühl, Hausle, Hill, Hille, Himwich, Huntington, Joffe, Kellogg, Kline, Langman, Harry Levy, Mirick, H. H. Mitchell, C. L. E. Moore, Mullins, Pell, Post, Rainich, Reddick, R. G. D. Richardson, Ritt, Seely, H. D. Thompson, Veblen, H. E. Webb, Wedderburn, Weiss, H. S. White, Whittemore, J. W. Young.

The Secretary announced the election of the following persons to membership in the Society:

Dr. Maria Castellani, Bryn Mawr College;
Miss Laura Guggenbühl, Bryn Mawr College;
Mr. Henry Alfred Hoover, Washington University;
Professor Benjamin Hoffman Kerstein, State Teachers' College Silver City, N. M.;
Professor James Shannon Miller, Emory and Henry College.

Thirty-seven applications for membership were received.

On recommendation of the Council, the following amendment to the By-Laws was presented to the Society: to insert in Section 1 of Article III the words "of ex-presidents,

for a period of six years after the expiration of their respective presidential terms," so that the section shall read:

"Section 1. The Council shall consist of the officers specified in Article I, of ex-secretaries who have served as Secretary for ten years or more, of ex-presidents for a period of six years after the expiration of their respective presidential terms, and of fifteen additional elected members chosen in accordance with Article IV, Section 2."

The Society voted to adopt this amendment.

At the meeting of the Council, the appointment of the following committees was announced: On arrangements for the Annual Meeting of 1924, to be held at Washington: Professors H. L. Hodgkins (chairman), J. T. Erwin, Archibald Henderson, A. E. Landry, F. D. Murnaghan, and J. M. Page, Miss Elizabeth W. Wilson, and the Secretary; on the Second Award of the Bôcher Prize: Professors G. D. Birkhoff (chairman), L. E. Dickson, and H. S. White.

The report of the Auditing Committee, appointed to examine the accounts of the Treasurer, was presented to the Council for its information; this report shows a balance for the general funds of the Society, as of December 15, 1923, of $9,638.78, of which $5,726.91 is held as reserve to protect life memberships and $2,500 is estimated as required for printing the outstanding numbers of the BULLETIN and TRANSACTIONS for 1922 and 1923.

The Committee on the next Colloquium reported a recommendation that it be held at Cornell University in the summer of 1925; this recommandation was adopted.

President Veblen presided at the scientific sessions, relieved by Professors E. W. Brown, H. S. White, and Vice-President J. H. M. Wedderburn. Titles and abstracts of the papers read at this meeting follow below. Mr. Van de Walle was introduced be Professor Huntington. Professor Murnaghan's paper was read by Mr. Rainich, and Mr. Van de Walle's by Professor Huntington. The papers of Alexander, Garabedian, Michal, Moore, Murray, Rainich (third paper), and Stone, were read by title.

1. Professor F. D. Murnaghan: *The outer multiplication of integral forms.*

The author gives a short proof by means of tensor algebra of a known theorem on the composition of integral forms. In any space S_n, not necessarily metrical, of n dimensions, the generalised Kronecker symbol $\delta^{r_1 \cdots r_p}_{s_1 \cdots s_p}$ $(p \leq n)$ may be defined by the following properties. It is alternating in both the upper labels $r_1 \cdots r_p$ and the lower labels $s_1 \cdots s_p$. It has the value zero if the group $r_1 \cdots r_p$ is different from the group $s_1 \cdots s_p$ and has the value ± 1 according as the arrangement $r_1 \cdots r_p$ is of the same class as the arrangement $s_1 \cdots s_p$ or not. The symbol $\delta^{r_1 \cdots r_p}_{s_1 \cdots s_p}$ defines a mixed tensor of rank $2p$. If we have two alternating covariant tensors $A_{l_1 \cdots l_i}$ and $B_{m_1 \cdots m_j}$ of ranks i and j respectively $(i+j \leq n)$ then $\delta^{l_1 \cdots l_i \, m_1 \cdots m_j}_{\alpha_1 \cdots \alpha_i \, \beta_1 \cdots \beta_j} A_{l_1 \cdots l_i} B_{m_1 \cdots m_j}$ is an alternating covariant tensor of rank $i+j$. If we construct integral forms with the components of the alternating covariant tensors as coefficients, the form of rank $i+j$ is the outer product of the two forms of rank i and j respectively. It may be expressly mentioned that the tensor $\delta^{r_1 \cdots r_p}_{s_1 \cdots s_p}$ is non metrical, i.e., the S_n need not be Riemannian nor need it have a geometry of paths. Thus the usual presentation, in the special case $p = 1$, of $\delta^{r_1}_{s_1}$ as equal to $g^{r_1 k} g_{s_1 k}$ is unfortunate.

2. Mr. G. Y. Rainich: *A modification of tensor analysis.*

In this paper, vectors are represented by their components with respect to a special system of rectangular axes at each point, and points, as usual by n coordinates. The derivative of a tensor φ of rank zero is given by $(\partial \varphi / \partial x\rho) \cdot b_{\rho i}$, where $b_{ij} = b_{ji}$, and the derivative of a tensor ψ_i of the first rank by $(\partial \psi_i / \partial x\rho) \cdot b_{\rho j} + \psi_\rho \, T^\rho_{ij}$, etc. If we require that (1) the derivatives of the Galilean tensor δ^i_j be zero and (2) the second derivatives of a tensor of rank zero be independent of the order of differentiation, we find that T^i_{jk} is one-half the determinant made up of three rows of three terms each: $\partial a_{i\rho} / \partial x\sigma, \cdots$; $b_{i\rho}, \cdots$; $b_{i\sigma}, \cdots$; where the a_{ij} are the coefficients of the inverse transformation with respect to the b_{ij}. There is no distinction between covariant and contravariant quantities. This representa-

tion and the usual one are connected by the formula $g_{ij} = a_{i\rho} \cdot a_{\rho j}$. Greek letters indicate a summation.

3. Mr. G. Y. Rainich: *On the electromagnetic field in general relativity theory.*

If at each point two perpendicular planes are given, we say we have a skeleton. A skeleton defines at each point two vectors p and q. The principal planes of the tensors of an anti-symmetrical field constitute a skeleton; in order that an anti-symmetrical field satisfy the Maxwell equations it is necessary and sufficient that the vectors p and q be gradients. If we have a Maxwellian field a necessary and sufficient condition that the planes of its skeleton may be considered as tangent planes to two families of surfaces is that the vector q be zero. The known energy relation which connects the contracted curvature tensor F_{ij} with the electromagnetic tensor imposes on F_{ij} the condition that the secular equation formed with the F_{ij} have two positive and two negative roots all of the same absolute value. Geometrically it means that F_{ij} defines a skeleton. The vector p of this skeleton is a gradient as a consequence of the general properties of curved space, but the condition that q be a gradient seems to impose a new condition on the space.

4. Professor R. E. Gilman: *The remainder in Charlier's integral formulas.*

The formula

$$\int_a^b \lambda(x)dx = \omega \sum_{r=1}^{s-1} \lambda(a+r\omega) + \tfrac{1}{2}\omega[\lambda(b)-\lambda(a)] - \frac{\omega}{12}[\Delta\lambda(b)-\Delta\lambda(a)] + \frac{\omega}{24}[\Delta^2\lambda(b)-\Delta^2\lambda(a)] + \cdots \cdots + C_n\omega[\Delta^n\lambda(b)-\Delta^n\lambda(a)] + R^n$$

is typical of those considered. In this formula, $\lambda(x)$ is a single-valued function with a continuous nth derivative in the interval $a \leq x \leq b + n\omega$, s is an integer, $\omega = (b-a)/s$, and $\Delta\lambda(a)$ has the customary meaning defined by the relations $\Delta\lambda(a) = \lambda(a+\omega) - \lambda(a)$ and $\Delta^{k+1}\lambda(a) = \Delta(\Delta^k\lambda(a))$. An investigation of the remainder R_n is carried out from two different points of view. The principal result of the paper is that

$$R_n = \omega^{n+1}(b-a)\lambda^{(n+1)}(\xi)\int_0^1 \frac{t(t-1)\dots(t-n)}{(n+1)!}dt, \qquad a < \xi < b+n\omega.$$

5. Professor Einar Hille: *An existence theorem.*

The integral equation of the Volterra type

$$w(z) = w_0(z) + \int_z^\infty \sin(t-z)\Phi(t)w(t)dt$$

is studied, in this paper, for complex values of z. The functions $w_0(z)$ and $\Phi(z)$ are supposed to be single-valued and analytic in the strip $A < x$, $B_1 < y < B_2$ $(z = x+iy)$ and to be bounded in absolute value; $\Phi(z)$ in addition satisfies $|\Phi(z)| < M/|z|^{1+\nu}$, $\nu > 0$. The path of integration is a straight line parallel to the real axis. The existence is shown of a unique solution which is single-valued and analytic in the strip. The continuation of the solution outside of the strip in a sector is studied in the case in which $w_0(z) = c_1e^{iz} + c_2e^{-iz}$.

6. Professor E. W. Brown: *A proof of the instability of the motions of asteroids whose periods are half that of Jupiter.*

This paper consists mainly of an announcement of a proof of the instability of the motions of asteroids which have half the period of Jupiter. The well known gap in the distribution corresponding to this period has long been a puzzle. It appears that the instability is mainly one which causes eccentricities to increase so much that the configurations cannot be maintained.

7. Dr. C. A. Garabedian: *Thick rectangular plates.*

The author has recently presented a method of series in elasticity* in connection with its applications to *Circular plates of constant or variable thickness*† and to *Rods of constant or variable cross-section.*‡ When the method is applied to rectangular plates the differential equations are partial instead of total, and the difficulties of integration are such that it is natural in a first presentation to avoid the complicated case of variable thickness. In the present note

* This BULLETIN, April and May, 1923; COMPTES RENDUS, vol. 177 (1923), p. 942.

† TRANSACTIONS OF THIS SOCIETY, July, 1923.

‡ This paper will appear in an early number of the AMERICAN JOURNAL.

the method of attack is illustrated by a computation of the exact solution for a rectangular plate of constant thickness, supported at the edges and under uniform pressure. The results will be found in the COMPTES RENDUS in a note entitled *Plaques rectangulaires épaisses.* The author will show later how the method of series applied twice (secondly to the auxiliary partial differential equations which present themselves in the plate problem) leads to total differential equations and furnishes new solutions for beams of rectangular section.

8. Dr. F. H. Murray: *Generalization of certain theorems of Bohl.*

The object of this paper is the generalization of certain existence theorems obtained by P. Bohl, concerning trajectories which remain in the neighborhood of a point of equilibrium for a one-sided variation of the time, or for all values of the time. When the equations in generalized coordinates of a dynamical problem are satisfied by a set of constant values of the variables, the theorems of Bohl cannot always be applied; in certain cases, however, similar theorems can be obtained by similar methods, and in this article a number of these are derived. They include the corresponding theorems of Bohl as special cases. (See P. Bohl, *Über die Bewegung eines mechanischen Systems in der Nähe einer Gleichgewichtslage,* CRELLE'S JOURNAL, vol. 127, pp. 179–276.)

9. Mr. A. D. Michal: *Functionals of closed plane curves, invariant under one-parameter groups of transformations of the plane, and generalizations.*

In a previous communication presented to the Society (Annual Meeting, 1923) the author commenced the study of functionals of closed plane curves invariant under a given arbitrary continuous one-parameter group of transformations of the plane. In the present paper the writer continues his discussion of such invariant functionals of closed plane curves. Furthermore the theory is extended to the case of functionals of closed plane curves depending in particular on a point on the curve. Generalizations of this invariant theory are effected in the case of functionals of closed $n-1$ spreads in n dimensions.

10. Mr. M. H. Stone: *An unusual type of expansion problem.*

In the present paper, the boundary value and expansion problems arising from the differential systems $u'+\lambda pu=0$, $u(0)-u(1)=0$, $0\leqq x\leqq 1$, and $u'+\lambda pu=0$, $u(0)+u(1)=0$, $0\leqq x\leqq 1$, are discussed. The function $p(x)$ is bounded, and is so restricted that the Lebesgue integral $P(x)=\int_0^x p dx$ has only a finite number of oscillations. The special case when p is unity is taken up first. The general case is then considered. When P is not monotone, the formal expansions of an arbitrary function represent, not the function itself, but a related function which may, in special cases, coincide with it. Infinitely many functions have the same formal expansions in terms of solutions of the two differential systems.

11. Mr. M. H. Stone: *On the order of an analytic function at a singular point.*

Hadamard in his Thèse defined the order of an analytic function at a singular point (JOURNAL DE MATHÉMATIQUES, ser. 4, vol. 8 (1892), pp. 101–186, part III). He did not prove the existence of the order. Later writers on the subject have neglected to give proofs, or have given incorrect proofs. In the present paper a demonstration of the existence of the order is given.

12. Professor J. R. Kline: *Theorems concerning the division of a plane by continua.*

Suppose the bounded decomposable continuum M, lying in the plane S, is such that if g is any closed connected subset of M, then $M-g$ is connected. Under these conditions, the author proves that M must divide the plane. He further proves that unless M is a simple closed curve it cannot be a continuous curve.

13. Professor L. P. Eisenhart: *Geometries of paths for which the equations of the paths admit a quadratic first integral.*

A geometry of paths for a general space of n dimensions is based upon the conception that the paths are fundamental

entities. They are the integral curves of a system of differential equations of the form

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0,$$

where s is a special parameter for each path and the $\Gamma^i_{\alpha\beta}(=\Gamma^i_{\beta\alpha})$ are functions of the x's. The present paper determines the functions $\Gamma^i_{\alpha\beta}$ in order that the above equations may admit a given quadratic first integral $g_{ij}(dx^i/ds)(dx^j/ds) = \text{const.}$ Veblen and Thomas have solved the converse problem: Given the functions Γ, to determine whether the equations admit a quadratic first integral.

14. Mr. W. E. Van de Walle: *On the complete independence of the postulates for betweenness.*

Of the eleven sets of postulates for betweenness published by Huntington and Kline in 1917 (TRANSACTIONS OF THIS SOCIETY, vol. 18, pp. 301–325), it is shown that sets 1–10 are completely independent, while set 11 is not. In the case of sets 1, 2, 3, 5, 6, 7, the necessary examples are given in terms of a class containing only four elements. In the case of sets 4, 8, 9, 10, 11, some of the examples require the use of a class containing five elements. The failure of set 11 to be completely independent is due to the non-existence of an example having the following "record": $A+$, $B+$, $C+$, $3-$, $4-$, $8+$.

15. Professor J. F. Ritt: *Equivalent rational substitutions.*

This paper presents several results relative to systems of three rational functions $\varphi(z)$, $\alpha(z)$, $\beta(z)$, for which $\varphi[\alpha(z)] = \varphi[\beta(z)]$.

16. Professor J. F. Ritt: *New proofs of two well known theorems on quadratic forms.*

Let N represent a prime positive integer. It is well known that if $N \equiv 1$, mod 4, it has the form $\alpha^2+\beta^2$, where α and β are integers, and that if $N \equiv 1$, mod 6, it is of the form $\alpha^2+\alpha\beta+\beta^2$. The present note proves these results by an analytic method, which involves the construction of a Riemann surface, and its uniformization with elliptic functions. It is not believed that such a method has been used before in the theory of numbers.

17. Professor R. L. Moore: *A characterization of a bounded continuum which forms the common boundary of two domains in the plane.*

It is shown that if, in a plane S, M is a bounded continuum which has more than one prime part and no one of its prime parts separates S then in order that $S-M$ should be the sum of two mutually exclusive domains such that every point of M is a limit point of both of them it is necessary and sufficient that M should contain no subcontinuum whose omission disconnects M.

18. Professor R. L. Moore: *On the relation of a bounded continuum to its complement in the plane.*

In a plane $\overline{S}$ let $\overline{M}$ denote a bounded continuum no one of whose prime parts separates $\overline{S}$. Let the term element denote any prime part of $\overline{M}$ or any point which does not belong to $\overline{M}$. Let S denote the set of all elements and let N denote the set of all prime parts of $\overline{M}$. An element p is said to be a limit element of a set of elements K if for every positive ϵ there exists an element belonging to K but distinct from p and containing a point at a distance less than ϵ from some point of p. It is shown that every "simple closed curve" of elements separates S into two "domains," of which a particular one is called its interior. It is shown that if the interior of every simple closed curve of elements is called a region then all the axioms of the author's *Foundations of plane analysis situs** are satisfied. It follows† that between the elements of S and the points of $\overline{S}$ there exists a one to one correspondence which preserves limits. The image $(\overline{N})$ of N under this correspondence is a continuous curve. Thus, from the stand-point of analysis situs, not only is N a continuous curve (of elements) but its relation to the remainder of the plane is that of a continuous curve to its complement.

19. Professor J. W. Alexander: *Note on Brouwer's fixed point theorem.*

Consider a continuous transformation T_1 of an n-sphere into itself such that no point is carried into its opposite.

* TRANSACTIONS OF THIS SOCIETY, vol. 17 (1916), pp. 131–164.
† Ibid., vol. 20 (1919), pp. 169–178.

Each point P may then be joined to its image P' by an arc of a great circle shorter than half a great circle; consequently, the transformation may be generated by a (singular) deformation and must be of index unity. Now, consider a continuous transformation T_2 with no fixed point. If T_2 be followed by a reflection, we obtain a transformation of the type T_1 and index one. The reflection is of index ± 1 depending on whether the sphere is a manifold of an odd or even number of dimensions. Therefore, the index of T_2 must be ± 1 depending on the dimensionality of the sphere.

20. Mr. G. Y. Rainich: *On the dependence of the curvature upon the electromagnetic field.*

Einstein introduces in his paper *Bietet die Feldtheorie Möglichkeiten für die Lösung des Quantumproblems?* (BERLIN SITZUNGSBERICHTE, December 13, 1923) a certain expression $\mathfrak{P}_{ik,lm}$ which has the algebraic properties of the Riemann tensor and reduces to the energy tensor after contraction. If we designate by $\Omega_{ik,lm}$ the tensor which depends in the same way on the dual (magnetoelectric) tensor as $\mathfrak{P}_{ik,lm}$ depends on the electromagnetic tensor, we have in $\Omega_{ik,lm}$ another expression with the same properties. We can form an infinity of linear combinations of these two tensors having the same properties. The expression proposed by the author in the December number of the PROCEEDINGS OF THE NATIONAL ACADEMY is a particular case of this, being equal to half of the difference of these two tensors.

R. G. D. RICHARDSON,
Secretary.

THE EQUATION OF THE EIGHTH DEGREE*

BY A. B. COBLE

1. *Introduction.* The progress of mathematics as a whole is occasionally brought to our attention by the appearance of some notable book or memoir in which the resources of the subject are brought from various fields to bear upon a central problem. An early instance of this is the *Ikosaeder*[1] of Klein—the forerunner of the Klein-Fricke series. The first two chapters of this book furnish an introduction to group theory which is as yet unsurpassed. A later example is the book of Hudson on *Kummer's Quartic Surface.*[2] This surface, remarkable in itself, is more remarkable for the breadth of the theories which attach to it. Central problems in these two books are respectively the quintic and the sextic equation. It is true that Hudson assumes the solution of the sextic when needful, but his book furnishes the geometric background for this solution.

Around the sextic equation there cluster the geometric theory of the Weddle and Kummer surfaces and of certain modular three-ways in S_4 as well as the analytic theory of the hyperelliptic integrals and functions of genus two, a complex in which the methods of algebra and group theory have constant play. For the most part the allied geometry is within the spaces of our experience, and with the correspondingly small number of variables the methods of ordinary analytic geometry are effective. The analytic function theory is also fairly manageable by direct methods. When, however, we seek to extend this field of ideas to equations of higher even degree complications develop

* Address as retiring chairman of the Chicago Section of this Society, presented to the Society at the meeting in Cincinnati, December 28, 1923. This is an outline of an investigation pursued under the auspices of the Carnegie Institution of Washington, D. C. References to a bibliography at the close of this paper are indicated by arabic numerals.

rapidly. The surfaces or manifolds involved lie in spaces of high dimension and are defined by many rather than one equation. The analytic theory is correspondingly difficult. Some solid ground there is, particularly in the theta functions, but not enough to form a basis for a closely knit theory.

It is the purpose of this paper to exhibit certain points of view, certain modes of approach to the general problem, and certain analytic methods which can be applied successfully to the octavic and in most cases to equations of even degree, $2p+2$. An indication there will be of a general theory which is fairly coherent and perhaps not more vague than is inevitable with generalizations of this character.

It should be remarked that the solutions here sought for these equations are solutions in terms of hyperelliptic modular functions and not the solutions of Klein in terms of the form problems of linear groups, which perhaps are more desirable but certainly are less accessible.

2. *Two Linear Systems of Irrational Invariants.* (*a*). The given equation defines three sets of points each set on a rational norm-curve in its space. Thus the sextic defines the set Q_6^1 of points $q_1, \ldots, q_6$ on a line S_1, the set R_6^2 of points $r_1, \ldots, r_6$ on a conic N^2 in S_2, and the set S_6^3 of points $s_1, \ldots, s_6$ on a space cubic curve N^3 in S_3. Similarly the octavic defines sets of eight points, Q_8^1, R_8^3, S_8^5 on respectively the line S_1, the space cubic curve N^3 in S_3, and the rational norm quintic curve N^5 in S_5. In general the equation of even degree $2p+2$ defines the sets Q_{2p+2}^1 on S_1, R_{2p+2}^p on N^p in S_p, and S_{2p+2}^{2p-1} on N^{2p-1} in S_{2p-1}. We shall take all of these norm curves N^k in S_k in the simple parametric form, $x_i = t^i (i = 0, \ldots, k)$, and assume that the parameters $t_1, \ldots, t_{2p+2}$ of the points of each set are either the roots of the given equation or are projective to these roots. We observe that for the extreme sets Q_{2p+2}^1, S_{2p+2}^{2p-1} the norm curve requirement imposes no conditions on the points since on $k+3$ points in general position in

S_k there is a unique norm curve N^k. The intermediate set R^p_{2p+2} is special due to $(p-1)^2$ conditions that its points shall lie on N^p. The extreme sets also furnish one of the simplest examples of *associated sets* ([3]Theorem 10).

(*b*). Invariants of a set of points in S_k are formed from the determinants of the coordinates of $k+1$ points selected from the set, which are combined in such a way that each term is of the same degree in each point. This degree is the *degree* of the invariant. Such invariants are *rational* or *irrational* according as they are or are not unaltered under permutation of this points ([4]I § 3). The sets Q^1_{2p+2} and R^p_{2p+2} have linear invariants which are linear combinations with constant coefficients of products of the respective types

$$\text{(A):}\qquad (12)(34)\cdots(2p+1,\ 2p+2),\ (ij)=\begin{vmatrix} t_i & 1 \\ t_j & 1 \end{vmatrix};$$

$$\text{(B):}\qquad (12\cdots,\ p+1)(p+2,\cdots,\ 2p+2),$$

$$\text{(I)}\qquad (i_1 i_2\cdots i_{p+1})=\begin{vmatrix} t^p_{i_1} & t^{p-1}_{i_1}\cdots 1 \\ t^p_{i_2} & t^{p-1}_{i_2}\cdots 1 \\ \cdots\cdots\cdots \\ t^p_{i_{p+1}} & t^{p-1}_{i_{p+1}}\cdots 1 \end{vmatrix}.$$

The various products of either type are linearly related in many ways by the usual determinant identities. The set S^{2p-1}_{2p+2} contributes nothing new to this invariant theory since its determinants are proportional to the complementary determinants of its associated set Q^1_{2p+2} ([4]I (25)).

(*c*). Obviously if the roots t_i of the given equation are known the numerical values of the invariants (A) are easily found. Conversely if the numerical values of the invariants (A) are known the roots t_i of the given equation may be found by rational processes. Thus from (12) (34) (56) ... $(2p+1,\ 2p+2)$ and (14) (32) (56) ... $(2p+1,\ 2p+2)$ we find by division the double ratio $(13\,|\,24)$. If we take $t'_1,\ t_2,\ t'_3$ to be 0, 1, ∞ the values of these double ratios determine the remaining roots $t'_4,\ \ldots,\ t'_{2p+2}$ of an equation

projective to the given equation. The procedure for finding the transformation which converts the known roots t' into the required roots t of the given equation is a known exercise in binary forms. We shall therefore henceforth regard the determination of the values of the invariants (A) as a "solution" of the given equation.

(*d*). The numerical values of the invariants (B) are in immediate relation to hyperelliptic modular functions of genus p. The basis notation[5] for these is very convenient. We divide the numbers $1, 2, \ldots, 2p+2$ into equivalent complementary sets and name the half periods by an even number of subscripts as in $P_{1,2,\ldots,2k} = P_{2k+1,\ldots,2p+2}$; the even theta functions of first order by a number of subscripts of the form $p+1\pm 4k$; and the odd functions of the first order by a number of subscripts of the form $p+1\pm 2(2k+1)$, $(k=0, 1, \ldots)$. The simplest set of modular functions are the values of the even functions for the zero arguments. When, however, the functions are hyperelliptic all of these values vanish except the ones with $p+1$ subscripts ([6]pp. 456–464). The non-vanishing values are connected with the invariants (B) by the remarkable relations

$$(2)\quad \epsilon\cdot(1, 2, \ldots, p+1)(p+2, \ldots, 2p+2) = \vartheta^4_{1,\ldots,p+1}((0)),$$

where ϵ is a properly adjusted sign (for $p=2$, cf. [4]I, p. 194; for $p=3$, [5]p. 111).

(*e*). If then we find rational expressions for the invariants of type (A) in terms of those of type (B) we have a solution of the given equation in terms of hyperelliptic modular functions. I shall return in § 3 to a discussion of these relations. Meanwhile let us seek for a natural and easily extensible way of introducing the theta functions themselves into the discussion. For this purpose the set of points S^{2p-1}_{2p+2} plays the important role.

3. *The Generalized Weddle p-way and the Hyperelliptic Kummer p-way.* (*a*). Given a general set of $k+3$ points $s_1, s_2, \ldots, s_{k+3}$ in S_k we define, by isolating a pair of points

s_1, s_2, a Cremona involution I_{12} of order k, projective to the type $x_i' = 1/x_i$ $(i = 0, 1, \ldots, k)$ which interchanges the points s_1, s_2 and has F-points at the remaining $k+1$ points of the given set. By considering the effect of these involutions upon the directions about the points of the given set we find that they generate a group for which the following generational relations hold:

$$\begin{aligned} &I_{12}I_{13} = I_{13}I_{12} = I_{23}, \\ &I_{12}I_{34} = I_{13}I_{24} = I_{14}I_{23} \equiv I_{1234}, \\ &I_{1234}I_{56} = I_{1235}I_{46} = \cdots \equiv I_{123456}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{3}$$

Hence they generate an abelian group of order 2^{k+2}, G_2^{k+2}, all of whose elements are involutory. We observe, however, an essential difference between the odd and even values of k. When k is odd and therefore $k+3$ even, say $2p+2$, there is one element of the group, namely

$$I = I_{123\ldots,\,2p+2} \tag{4}$$

which is symmetrically related to the given set of $2p+2$ points, S_{2p+2}^{2p-1} (cf. [4]II, p. 369). We define the Weddle p-way, W_p, in S_{2p-1}, to be the locus of fixed points of the involution I. That there is such a locus of fixed points and that it can be expressed parametrically by means of hyperelliptic functions we proceed to prove by a process which is new to the literature.

(*b*). The two common canonical forms of the hyperelliptic algebraic plane curve of genus p are first the form

$$y^2 = a_0(x-t_1)\ldots(x-t_{2p+2}), \tag{5}$$

commonly used for the development of the hyperelliptic integrals; and secondly the curve H_p of order $p+2$ with a p-fold point at O and $2p+2$ contacts $\tau_1, \tau_2, \ldots, \tau_{2p+2}$ of tangents from O. To the so-called "superposed points" x, $\pm y$ of (5) there correspond the pairs of points on H_p on a line through O. These pairs constitute the unique g_1^2 on H_p invariant under all birational transformations. The form (5) depends upon $2p-1$ "moduli" or "birational in-

variants". On the other hand the curve H_p has $3p-1$ absolute projective constants. Hence the ∞^{3p-1} projectively distinct curves H_p divide into ∞^{2p-1} birationally distinct classes, each class containing ∞^p curves. The ∞^p curves H_p in each class are birationally equivalent and it may be shown that the curves of a given class may all be obtained from a given curve of the class by selecting on this given curve a p-ad of points and transforming this p-ad into the p-fold point of the transform. This may in fact be done by Cremona transformations of the entire plane. However, each selected p-ad determines its "superposed" p-ad and either one of a pair of superposed p-ads leads to the same transformed curve. Thus the two aggregates—(1) the ∞^p projectively distinct but birationally equivalent curves H_p; and (2) one of the curves H_p and the ∞^p pairs of superposed p-ads on it—are birationally equivalent. In the next section we connect the first aggregate with the ∞^p points on the Weddle W_p; and in the following section the second aggregate with the variables of the attached theta functions and thereby obtain the desired parametric expression of W_p.

(*c*). Each of the ∞^p curves H_p has a p-fold point O and $(2p+2)$ branch points $\tau_1, \tau_2, \cdots, \tau_{2p+2}$ which make up a set of points, T^2_{2p+3}. The associated set lies in S_{2p-1} and by virtue of a property ([3]*p*. 1) of such associated sets the points in S_{2p-1} which correspond to $\tau_1, \cdots, \tau_{2p+2}$ can be identified with the fixed set S^{2p-1}_{2p+2}. Then as H_p varies in the ∞^p system the point P in S_{2p-1} which corresponds to O in the plane runs over a p-way in S_{2p-1}. In order to identify this p-way with the Weddle W_p we observe that H_p is a locus of fixed points of a Jonquière Cremona involution J_p and therefore the set T^2_{2p+3} is congruent to itself under J_p. Moreover J_p is a product of Jonquière quadratic involutions $J_0(O, \tau_1, \tau_2)$, $J_0(O, \tau_3, \tau_4), \cdots$. Hence (cf. [4]II, p. 361) the associated set in S_{2p-1} is congruent to itself under the product of the involutions $I_{12}, I_{34}, \cdots$, i. e. under I. Since the

F-points of I are all in the set S^{2p-1}_{2p+2} the point P is a fixed point of I and therefore a point of W_p. Conversely for each point P on W_p we have an associated set T^2_{2p+3} and a curve H_p. The known[7] relations satisfied by the set T^2_{2p+3} lead to geometric conditions on P. Thus for $p=2$ the six branch points τ are on a conic and therefore P is the vertex of a quadric cone on S^3_6— the usual definition of W_2. When $p=3$ the eight points τ and O are the base points of a pencil of cubics, whence P and the 8 points of S^5_8 are on a pencil of elliptic sextics in S_5 ([3]pp. 17–18).

(*d*). Consider now a fixed curve H_p and a p-ad of points on it which correspond to the points $(x_1), \ldots, (x_p)$ on (5). On the Riemann surface of (5) select p paths of integration from the fixed branch point t_1 to $(x_1), \ldots, (x_p)$ respectively. If v stands in succession for one of the p normal integrals of the first kind and we set

$$v_{t_1}^{(x_1)} + \cdots + v_{t_1}^{(x_p)} = u, \tag{6}$$

then to within multiples of the periods the p-ad defines a value system $u \equiv u_1, \ldots, u_p$ and conversely u defines as a solution of the inversion problem a p-ad $(x_1), \ldots, (x_p)$. Superposed p-ads give rise to $\pm u$. The particular p-ad at the point O is determined by $u = d$. In this way to a point P on W_p we attach parameters $\pm d$. We readily find (the exposition in [6]Chap. 10 is convenient here) that the involutions I_{12}, which in the planar field are represented by $J_0(O, \tau_1, \tau_2)$, transform W_p into itself and are given in terms of the parameters by the simple relations

$$d' \equiv d + \pi_{12} \tag{7}$$

where π_{12} is a particular half period. We still have to account, on the Weddle p-way, for the 2^{2p} singular points and the 2^{2p} singular tangent spaces of the Kummer p-way. It is more convenient to illustrate these by the relatively simple cases of the sextic and octavic for $p = 2, 3$ respectively.

(*e*). When $p=2$ the six branch points $\tau_1, \ldots, \tau_6$ of H_2 with node at O are on a conic and the discriminant con-

dition (cf. [8]§ 13) that τ_1, τ_2, τ_3 are on a line implies also that τ_4, τ_5, τ_6 are on a line. Then we find as in (d) that the even function $\Theta_{123}(d) = \Theta_{456}(d)$ vanishes. The corresponding condition on the associated set S_6^3, P of the set T_7^2 is that P is on a plane with s_1, s_2, s_3 and s_4, s_5, s_6 or that P is on a quadric section of W_2 by the pair of planes $\overline{s_1\, s_2\, s_3}, \overline{s_4\, s_5\, s_6}$. On transforming this condition on T_7^2 by $J_0(O, \tau_1, \tau_2)$; on S_6^3, P by I_{12}; and on $\Theta_{123}(d)$ by (7) we find a second type of discriminant condition which is satisfied when τ_3 approaches coincidence with O; when P approaches coincidence with s_3 in a direction on the quadric cone with node at s_3 and simple points at the other points of S_6^3, or P is on the quadric section of W_2 by this cone; and finally when the odd function $\Theta_3(d)$ vanishes. In this fashion we conclude that the quadric sections of W_2 by quadrics on S_6^3 are represented by the plane sections of the surface K_2 obtained by setting the coordinates equal to four linearly independent theta squares, i. e. of the Kummer surface. If four points $\tau_1, \ldots, \tau_4$ are on a line, $d \equiv \pi_{56}$, one of the fifteen proper half periods, and the point P of W_2 is on the line $\overline{s_5\, s_6}$. Finally if P is on the cubic curve N^3 through S_6^3 the set T_7^2 is on a conic ([3]Theorem 10) and H_2 is this conic doubly covered while $d \equiv 0$. In fact $d \equiv 0$ implies that a pair of the g_1^2 are nodal parameters which can happen only for a doubly covered conic for which the pairs of g_1^2 are the doubly covered points and O is any of these double points. This is of course the indeterminate case for the inversion problem.

When $p = 3$ the four linearly independent quadrics on S_6^3 above are replaced by $8 = 2^p$ linearly independent cubic spreads with nodes at S_8^5 and containing the norm curve N^5 on S_8^5; and these map W_3 on the generalized Kummer K_3 of hyperelliptic type. The $2^{2p} = 64$ singular tangent spaces arise from the 8 cubic spreads (cones) with respectively a triple point at one of the eight points of S_8^5 and double curve N^5; and the 56 degenerate cubic spreads which break up into an S_4 on five of the eight points and

a quadric on these five with nodes at the other three. These are connected with the 64 theta squares as above.

(*f*). Naturally we must have, in order to obtain specific analytic results such as (2), some analytic method for the hyperspace cases. Since the given equation defines in S_{2p-1} the set S^{2p-1}_{2p+2} and the norm curve N^{2p-1} on it, or if one prefers the N^{2p-1} with the set S^{2p-1}_{2p+2} on it, it is convenient to refer the points of the space to this norm curve. The case $p = 2$ is perhaps too special to be fairly illustrative, but I cannot go into the higher cases here. Let the parameters t of the six points of S^3_6 on N^3 be given by the binary sextic $(\alpha t)^6 = 0$; and let the three osculating planes of N^3 which meet in a point a of S_3 be given by the binary cubic

$$(at)^3 = (a't)^3 = \cdots = 0.$$

Then, if

$$(\beta t)^2 = \beta_0 t_1^2 + 2\beta_1 t_1 t_2 + \beta_2 t_2^2$$

is an arbitrary quadratic, β_3 an arbitrary constant, we have, in

$$\beta_3(\alpha a)^3(\alpha a')^3 + (aa')^2(a\beta)(a'\beta) = 0, \tag{8}$$

for variable β_3, β_0, β_1, β_2 the web of quadrics on S^3_6 in variables a_0, a_1, a_2, a_3, the coefficients of the variable cubic $(at)^3$. The polarized form of (8) in variables a, b is

$$\beta_3(\alpha a)^3(\alpha b)^3 + (ab)^2(a\beta)(b\beta) = 0. \tag{9}$$

If we assume that the quadric (8) has a node at a, then in (9) the coefficients of b_0, b_1, b_2, b_3 vanish giving rise to four equations bilinear in a and β. If from these we eliminate the β's we have the quartic equation in variables a of the Weddle surface W_2. If, however, we eliminate the a's we have the quartic equation in variables β of the Kummer surface K_2 as an envelope. H. F. Baker has given these equations as four-row determinants ([9]p. 65 and p. 56). Baker remarks ([9]p. 60) that K is a simultaneous invariant of the quadratic $(\beta t)^2$ and the sextic $(\alpha t)^6$. Strictly speaking this is not so, because of the non-homogeneity in the coefficients. However K can be written

$$K = K_0\beta_3^4 + K_1\beta_3^3 + K_2\beta_3^2 + K_3\beta_3 + K_4, \tag{10}$$

where the K_i are such simultaneous invariants. From the degrees and to some extent from the form we infer at once that these coefficients must be numerical multiples of respectively: the catalecticant of $(\alpha t)^6$; the apolarity invariant of $(\beta t)^2$ and the quadratic covariant of $(\alpha t)^6$ of degree 3; the apolarity invariant of $[(\beta t)^2]^2$ and $(\alpha\alpha')^4(\alpha t)^2(\alpha' t)^2$, together with the product of $(\beta\beta')^2$ and $(\alpha\alpha')^6$; the apolarity invariant of $(\alpha t)^6$ and $[(\beta t)^2]^3$; and $[(\beta\beta')^2]^2$. Similarly the surface W is a simultaneous invariant of $(\alpha t)^6$ and $(at)^3$ which from the degrees must evidently be the apolarity invariant of $(\alpha t)^6$ with the product of $(at)^3$ and its cubic covariant, i. e.

$$(11) \qquad W \equiv (aa')^2(a'a'')(a\alpha)(a''\alpha)^2(a'''\alpha)^3 = 0.$$

(*g*). The extension of this method to further cases constitutes a remarkably neat application of many phases of the theory of binary forms. Let me give one illustration. For $p = 3$ and given S_8^5, a point in S_5 is determined by a binary quintic $(at)^5 = (a't)^5 = \cdots$. The catalecticant of this quintic,

$$(aa')^2(aa'')^2(a'a'')^2(at)(a't)(a''t) = 0,$$

determines a one-parameter (t) system of cubic spreads with N^5 as a double curve and triple point at t. Thus for $t = t_1, \ldots, t_8$ we have respectively the eight cubic cones mentioned in 5. The existence of this system leads to the following theorem.

THEOREM. *The Kummer 3-way in S_7 with 64 singular points and 64 singular tangent spaces has in the hyperelliptic case an additional configuration of 64 S_3's each with the following property: An S_3 contains 8 singular points and is contained in 8 singular tangent spaces. The eight points lie on a cubic curve in the S_3 with parameters $t_1, \ldots, t_8$ and they lie three at a time on the remaining 56 singular tangent spaces.*

We observe then that the generalization of the conics in the singular tangent planes of the Kummer K_2 is in

part the 2-ways in the singular tangent sections of K_3, and in part the cubic curves in the 64 S_3's, each curve containing an intermediate set R_6^3.

A fuller development of these ideas leads quite naturally to the proof of the formula (2) which connects the invariants (B) and the theta fourth powers. To these modular relations I shall briefly return.

4. *The Modular Spreads Associated with the Invariants* (A) *and* (B). (*a*). The number of invariants (B) is $\frac{1}{2}\binom{2p+2}{p+1}$. I have shown ([4]I(10)) that only $\nu = \frac{1}{p+2}\binom{2p+2}{p+1}$ of these are linearly independent. Similarly the number of invariants (A) is $(2p+2)!/(p+1)!\,2^{p+1}$. I shall show in a forthcoming paper which will elaborate the indications presented here that the number of linearly independent invariants (A) is likewise ν. Thus the invariants (A) and (B) lie in linear systems of the same dimension $\nu-1$ and it is sufficient to know the values of ν independent invariants of either system in order to obtain all of them. They are of course subject also to many relations of higher degree. By means of either linear system the totality of birationally distinct binary $(2p+2)$-ics is mapped upon the points of a "modular spread" M_{2p-1} of dimension $2p-1$ in $S_{\nu-1}$.

(*b*). For the invariants (A) we secure the mapping by fixing all but one of the points of the set S_{2p+2}^{2p-1} at a convenient "base" and allowing the last one to vary over S_{2p-1}. Then the invariants (A) become spreads of order p with $(p-1)$-fold points at the fixed base points and this linear system maps the space S_{2p-1} upon M_{2p-1}. To the permutation group of the roots of the underlying binary $(2p+2)$-ic there corresponds, in $S_{\nu-1}$, a collineation group of order $(2p+2)!$, $C_{n!}$, under which M_{2p-1} is invariant; in S_{2p-1}, however, the Moore cross-ratio group which is generated by $(2p+1)!$ permutations of the fixed base points and an additional Cremona involution with $2p$ of the fixed base points as F-points and the remaining one a fixed point (cf. for the sextic [10]§ 1).

For the invariants (B) attached to the set R^2_{2p+2} we select ν independent ones as coordinates in $S_{\nu-1}$ and then find that the permutation of the roots or of the points of the set R leads likewise to a collineation group $C'_{n!}$ in $S_{\nu-1}$. It is the obvious thing to make such a selection of coordinates that the collineation groups $C_{n!}$ and $C'_{n!}$ attached respectively to the systems (A) and (B) shall coincide. Furthermore in order to distinguish between the two mappings we map by the system (A) upon the points of $S_{\nu-1}$ and by the system (B) upon the $S_{\nu-2}$'s in $S_{\nu-1}$.

(*c*). For the sextic, $\nu = 5$ and the modular spread M_3 in S_4 is a cubic three-way, the map of S_3 by quadrics on 5 points. By proper choice of the quadrics M_3 takes the form $\sum_{i=1}^{6} x_i^3 = 0$ where $\sum_{i=1}^{6} x_i = 0$. The collineation $C_{6!}$ is then merely the permutations of the six coordinates x_i. By proper choice of the invariants (B) the planar sets of 6 points on a conic, R^2_6, are mapped upon the S_3's of S_4, $u_1, \ldots, u_6$ where $\sum_{i=1}^{6} u_i = 0$ in such a way that the relation

$$(\sum u_1 u_2)^2 - 4(\sum u_1 u_2 u_3 u_4) = 0$$

is satisfied. Here then the mapping is such that the invariants (A), (B) of the same binary sextic give rise to point and tangent space at the point of the *same* modular spread ([10](33)).

For the octavic, $\nu = 14$, and the geometry is hardly suitable for hasty exposition. Thus the modular spread attached to the invariants (A) is an M_5^{40} of order 40 and dimension 5 in S_{13}. We have, however, in the formula

$$\varepsilon(ij)(kl)(mn)(op) = \vartheta^2_{ijkl}\vartheta^2_{ijmn}\vartheta^2_{ijop}, \tag{12}$$

where ε is a properly chosen sign, the expression of the invariants (A) as uniform modular functions which are in immediate algebraic relation with the invariants (B). For detailed study the tactical relations exhibited by E. H. Moore[11] are most advantageous.

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GROUP OF A SET OF SIMULTANEOUS ALGEBRAIC EQUATIONS*

BY LOUIS WEISNER

1. *Introduction.* Consider a system of m independent and consistent algebraic equations in m variables

$$f_k(z_1, \cdots, z_m) = 0, \qquad (k = 1, \cdots, m). \tag{1}$$

Let the roots of this system of equations be $(z_{1a}, \ldots z_{ma})$ $(a = 1, \cdots, n)$. We assume that each z_{pa} is finite. Therefore by a process of elimination we obtain equations

$$F_p(z_p) = 0, \qquad (p = 1, \cdots; m), \tag{2}$$

each of which is of degree n and involves only one of the unknowns. None of these equations vanishes identically, since equations (1) are independent. We assume that none of equations (2) has a multiple root.

A rational domain R which includes the coefficients of (1) includes the coefficients of (2). It will be shown in this paper that the groups of equations (2) relative to R are identical, except for the symbols which they affect.

2. *Elimination of z_m.* On eliminating z_m from (1) we obtain $m - 1$ equations involving $z_1, \cdots, z_{m-1}$. Now z_{m-1} cannot be absent from all of these equations; for in that case we would have $m - 1$ equations involving fewer than $m - 1$ unknowns, and equations (1) would not be consistent and independent as assumed. Repeating this argument we can find two independent and consistent equations

$$g_1(z_1, z_2) = 0, \qquad g_2(z_1, z_2) = 0, \tag{3}$$

involving the unknowns z_1, z_2 (or any other pair of the unknowns). This elimination can always be carried out in such a way that if (z_{1a}, z_{2a}) is a root of equations (3), then there is a root $(z_{1a}, z_{2a}, \cdots, z_{ma})$ of equations (1).

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3. *Relations of Rationality.* Let R' be the domain obtained by adjoining z_{2a} to R. By equations (3), $g_1(z_{1a}, z_{2a})$ and $g_2(z_{1a}, z_{2a})$ are rational functions of a root of $F_1(z_1) = 0$, with coefficients in R' which equal a number (zero) in R'; they are therefore unaltered in value by the substitutions of the group H of $F_1(z_1) = 0$ relative to R'. Suppose that a substitution of H changes z_{1a} to z_{1b}. Applying this substitution to $g_1(z_{1a}, z_{2a})$ and $g_2(z_{1a}, z_{2a})$ we obtain $g_1(z_{1b}, z_{2a}) = 0$ and $g_2(z_{1b}, z_{2a}) = 0$. By § 2, there are two roots $(z_{1a}, z_{2a}, \cdots, z_{ma})$ and $(z_{1b}, z_{2a}, \cdots, z_{mb})$ of equations (1) with the same value for z_2. Therefore $F_2(z_2) = 0$ has a multiple root, contrary to assumption. It follows that H leaves z_{1a} fixed. Hence z_{1a} is in R' and is a rational function of z_{2a} with coefficients in R.

THEOREM 1. *If $(z_{1a}, z_{2a}, \cdots, z_{ma})$ is a root of equations (1), z_{pa} equals a rational function of z_{qa} with coefficients in R $(p, q = 1, 2, \cdots, m)$*.*

4. *Group Relations.* Let us regard G_p, the group of $F_p(z_p) = 0$ relative to R, as a substitution group not on the roots $z_{p1}, z_{p2}, \cdots, z_{pn}$, but on their second subscripts. Thus (12) simultaneously interchanges z_{11} and z_{12}, z_{21} and z_{22} etc. With this understanding we state our main result.

THEOREM 2. *The groups of equations (2) relative to R are identical.*

We shall prove G_1 identical with G_2. By Theorem 1, $z_{pa} = \varphi_{pa}(z_{1a})$ where φ_{pa} is a rational function with coefficients in R. Hence by equations (1)

$$(4) \quad f_k[z_{1a}, \varphi_{2a}(z_{1a}), \cdots, \varphi_{ma}(z_{1a})] = 0, \quad \begin{pmatrix} k = 1, 2, \cdots, m \\ a = 1, 2, \cdots, n \end{pmatrix}$$

* The following is a proof which might be offered of this theorem: Eliminate every power of z_1 from equations (3) except the first; then z_1 is expressed as a rational function of z_2 with coefficients in R. The objection to this proof is that in the process of elimination the first power of z_1 may be incidentally eliminated. It is easy to make up examples where this actually occurs. I believe the proof given above is free from objection.

Now suppose a substitution of G_1 replaces z_{1a} by z_{1b}. Then equations (4) become

$$f_k[z_{1b},\ \varphi_{2a}(z_{1b}),\ \cdots,\ \varphi_{ma}(z_{1b})] = 0. \tag{5}$$

Therefore $[z_{1b},\ \varphi_{2a}(z_{1b}),\ \cdots,\ \varphi_{ma}(z_{1b})]$ is a root of (1). But $[z_{1b},\ \varphi_{2b}(z_{1b}),\ \cdots,\ \varphi_{mb}(z_{1b})]$ is a root of (1), and (2) has no multiple root. Hence $\varphi_{pa}(z_{1b}) = \varphi_{pb}(z_{1b})$. We may therefore omit the second subscript of φ_{pa} and write

$$z_{pa} = \varphi_p(z_{1a}). \tag{6}$$

Now let $\psi(z_{21}, \cdots, z_{2n})$ be a rational function with coefficients in R of the roots of $F_2(z_2) = 0$ which equals a number in R. Then by (6) $\psi[\varphi_2(z_{11}), \cdots, \varphi_2(z_{1n})]$ is a rational function with coefficients in R of the roots of $F_1(z_1) = 0$ which equals a number in R and hence is unaltered by G_1. But on applying a substitution of G to $\psi[\varphi_2(z_{11}), \cdots, \varphi_2(z_{1n})]$ we find that, in virtue of (6), $z_{21}, \cdots, z_{2n}$ undergo a similar substitution. Therefore $\psi(z_{21}, \cdots, z_{2n})$ is unaltered by every substitution of G_1. It follows that G_1 is a subgroup of G_2. Similarly G_2 is a subgroup of G_1. Hence G_1 and G_2 are identical.

5. *A Necessary and Sufficient Condition.* We shall call $G = G_p$ $(p = 1, \cdots, m)$ the group of equations (1) relative to R. Let $\psi(z_{11}, \cdots, z_{1n}; \cdots; z_{m1}, \cdots, z_{mn})$ be unaltered in value by all the substitutions of G. Then so is $\psi[z_{11}, \cdots, z_{1n}; \cdots; \varphi_m(z_{11}), \cdots, \varphi_m(z_{1n})]$. But the latter is a rational function with coefficients in R of the roots of $F_1(z_1) = 0$, which is unaltered by G. Hence it equals a number in R. Conversely, let $\psi(z_{11}, \cdots, z_{1n}; \cdots; z_{m1}, \cdots, z_{mn})$ equal a number in R. Then so does $\psi[z_{11}, \cdots, z_{1n}; \cdots; \varphi_m(z_{11}), \cdots, \varphi_m(z_{1n})]$. Since the latter is unaltered by G, so is the former.

THEOREM 3. *A necessary and sufficient condition that a rational function with coefficients in R, of the numbers satisfying a set of independent and consistent algebraic equations with coefficients in R, be unaltered in value by the group of the equations relative to R, is that it equal a number in R.*

THE UNIVERSITY OF ROCHESTER

ON A TYPE OF PLANE UNICURSAL CURVE

BY HAROLD HILTON

The type considered is that which meets the sides of a real triangle in points whose parameters have at most three distinct values. If we take the triangle as triangle of reference, and suitable homogeneous coordinates, we may suppose any point (x, y, z) on the curve given by the equations

$$(1) \qquad \begin{cases} x = (t-\alpha)^{p_1}(t-\beta)^{q_1}(t-\gamma)^{r_1}, \\ y = (t-\alpha)^{p_2}(t-\beta)^{q_2}(t-\gamma)^{r_2}, \\ z = (t-\alpha)^{p_3}(t-\beta)^{q_3}(t-\gamma)^{r_3}, \end{cases}$$

where

$$(2) \qquad p_1+q_1+r_1 = p_2+q_2+r_2 = p_3+q_3+r_3 = n,$$

where n is the degree of the curve.

In this, one of the quantities p_1, p_2, p_3 is zero and the other two are zero or positive integers; and so for q_1, q_2, q_3 and r_1, r_2, r_3.

We may denote the curve (1) symbolically by the array

$$(3) \qquad \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix},$$

or, as it may be written to save space,

$$(4) \qquad (p_1\ q_1\ r_1,\quad p_2\ q_2\ r_2,\quad p_3\ q_3\ r_3).$$

It is evident that (*a*) each row of (3) contains at least one zero, (*b*) the sum of the elements in each column is n, (*c*) no two columns are identical, (*d*) we get no essentially distinct curve, if we interchange two rows or interchange two columns.

The first problem that suggests itself is to find the number of essentially distinct curves of the required type; i. e., to find the number of distinct arrays (3) subject to the restrictions just mentioned. We content ourselves with giving the

result in the form of a table, the first column giving the array (4), and the last two giving the number of distinct curves when n is odd or even.

Note that in this table

$$(5)\qquad \begin{cases} p+P=q+Q=n, \\ \lambda+\mu+\nu=n, \\ f+F=g+G=h+H=n; \end{cases}$$

f is not less than F, g, G, h, or H.

Array		n odd	n even
$(n00,\ 0n0,\ 00n)$		1	1
$(pP0,\ 0n0,\ 00n)$		$n-1$	$n-1$
$(0pP,\ 0n0,\ 00n)$	$p \geqq P$	$\frac{1}{2}(n-1)$	$\frac{1}{2}n$
$(\lambda\mu\nu,\ 0n0,\ 00n)$	$\mu \geqq \nu$	$\frac{1}{4}(n-1)^2$	$\frac{1}{4}n(n-2)$
$(pP0,\ qQ0,\ 00n)$	$p \geqq Q;\ p>P, q$	$\frac{1}{4}(n-1)^2$	$\frac{1}{4}n(n-2)$
$(0pP,\ qQ0,\ 00n)$		$(n-1)^2$	$(n-1)^2$
$(\lambda\mu\nu,\ qQ0,\ 00n)$	$\lambda>\mu$ or $\lambda=\mu,\ q>Q$	$\frac{1}{4}(n-1)^2\,(n-2)$	$\frac{1}{4}n(n-2)^2$
$(fF0, f0F, 0Ff)$	$f>F$	$\frac{1}{2}(n-1)$	$\frac{1}{2}(n-2)$
$(fF0, F0f, 0fF)$	$f>F$	$\frac{1}{2}(n-1)$	$\frac{1}{2}(n-2)$
$(ff0,\ f0f,\ 0ff)$		0	1
$(fF0, f0F, 0hH)$	$f>F;\ f>h \geqq H$	$\frac{1}{6}(n-1)\,(n-3)$	$\frac{1}{6}n(n-2)$
$(fF0, F0f, 0hH)$		$\frac{1}{4}(n-1)\,(n-3)$	$\frac{1}{4}(n-2)^2$
$(fF0, g0G, 0Ff)$	$f>F;\ f>g \geqq G$	$\frac{1}{6}(n-1)\,(n-3)$	$\frac{1}{6}n(n-2)$
$(fF0, g0G, 0gG)$	$f>g, G;\ g \neq G$	$\frac{1}{4}(n-1)\,(n-3)$	$\frac{1}{4}(n-2)\,(n-4)$
$(fF0, g0G, 0Gg)$	$f>g, G;\ g \neq G$	$\frac{1}{4}(n-1)\,(n-3)$	$\frac{1}{4}(n-2)\,(n-4)$
$(fF0,\ g0g,\ g0g)$		0	$\frac{1}{2}(n-2)$
$(fF0, g0G, 0hH)$		$\frac{1}{6}(n-1)\,(n-3)\,(n-5)$	$\frac{1}{6}(n-2)\,(n-3)\,(n-4)$

There are in all $\frac{1}{12}n(5n^2+7)$ distinct curves, if n is odd, and $\frac{5}{12}n(n^2+2)$, if n is even. In this, however, may be included some curves of degree less than n if two rows of (3) are proportional. This can only be the case, if the curve belongs to the type $(0pP,\ 0n0,\ 00n)$ and p and n have a common factor, or to the type $(\lambda\mu\nu,\ qQ0,\ 00n)$ and q is a multiple of n', where $\nu/n = \nu'/n'$ and ν' is prime to n'.

For $n = 3$, there are 13 cubics, 4 cuspidal, 4 crunodal, and 5 acnodal.

For $n = 4$, there are 29 quartics, excluding (022, 040, 004), which is a conic. Of these, 8 have a superlinear branch of order three, 3 have a cusp-linear triple point, 6 have a rhamphoid cusp and another double point, 2 have a crunode and two acnodes, 2 have a cusp and two acnodes, 2 have two cusps and an acnode, 1 has three cusps, and 5 have three acnodes.

Among them we may note the following, whose equations are given referred to the triangle with its vertices at the double points:

(211, 040, 004), $3(y^2z^2+z^2x^2+x^2y^2)+2xyz(2\sqrt{2}\,x+2\sqrt{2}\,y+z)=0,$

(310, 130, 004), $(x^2-6xy+y^2)z^2+x^2y^2=0,$

(031, 220, 004), $z^2(x-y)^2-4xyz(x+y)+16x^2y^2=0,$

(211, 130, 004), $z\{3(x+y)+\sqrt{5}\,(x-y)\}\{(x+y)z+xy\}+2x^2y^2=0,$

(310, 202, 013), $y^2z^2+z^2x^2+x^2y^2+xyz(x-2y-2z)=0,$

(310, 202, 022), $y^2z^2+z^2x^2+x^2y^2-2xyz(x+y)=0.$

It will be noticed that the curve obtained by the quadratic transformation, which consists in replacing x, y, z by $1/x$, $1/y$, $1/z$ in (1), is a curve of the same type.

Without loss of generality we may suppose that in (1) the numerically greatest of p_2-p_3, $p_3-p_1, \ldots, r_1-r_2$ is r_3-r_1, and that $r_3 \geqq r_2 \geqq r_1 = 0$, $p_1-p_3 \geqq q_1-q_3$. We may then put (1) in the form

$$x = t^a(t-1)^b,\ y = t^c(t-1)^d,\ z = 1, \tag{6}$$

on replacing t by

$$\{\gamma(\beta-\alpha)t-\alpha(\beta-\gamma)\} \div \{(\beta-\alpha)t-(\beta-\gamma)\},$$

and replacing x, y, z by suitable multiples of themselves.

In (6) we have

$$a = p_1-p_3,\ b = q_1-q_3,\ c = p_2-p_3,\ d = q_2-q_3, \tag{7}$$

$$a \geqq b \geqq 0,\ (a+b) \geqq (c+d) \geqq 0,\ b+c \geqq 0,\ a+d \geqq 0, \tag{8}$$

in virtue of the restrictions which we have supposed to hold between $p_1, p_2, \ldots, r_3$.

The sides of the triangle of reference meet the curve only where $t = 0$, 1, or ∞. We shall suppose excluded the case $ad = bc$, when the curve becomes the well known type

$$x : y : z = t^a : t^c : 1.$$

The tangent to (6) is

$$(9)\quad t^{-a}(t-1)^{-b}\{(c+d)t-c\}\,x - t^{-c}(t-1)^{-d}\{(a+b)t-a\}\,y + \{(a+b-c-d)t-(a-c)\}\,z = 0.$$

The inflexions, other than $t = 0$, 1, ∞, are given by

$$(10)\quad \{(a+b-c-d)t-(a-c)\}\{(a+b)t-a\}\{(c+d)t-c\} + (ad-bc)t(t-1) = 0.$$

We see that the degree n of (6) is $a+b$ if $c \geqq 0$, $d \geqq 0$; $a+b-d$ if $c \geqq 0$, $d \leqq 0$; $a+b-c$ if $d \geqq 0$, $c \leqq 0$.

The class is in general the greatest of

$$a+b+1,\ a+d+1,\ b+c+1.$$

It may be less than this by unity in certain cases, e. g., if $a+b = c+d$.

The knowledge of the degree and class, combined with the fact that the deficiency is zero, gives the Plücker's numbers of the curve.

If ABC is the triangle whose sides are met by (6) in points whose parameters are 0, 1, ∞. we have from (9) and (10) the following result.

From each of A, B, C at most one tangent can be drawn touching the curve at a point not lying on a side of the triangle ABC, and the number of these tangents is equal to the number of inflexions which do not lie on the sides of the triangle.

This number is three at most, but can be two, one, or zero, if one or more of

$$a+b-c-d,\ c+d,\ b,\ c,\ d,\ a-c,\ b-d$$

is zero. For instance, the type ($ff0$, $f0f$, $0ff$), in which $c+d$, b, and $a-c$ are zero, has no inflexion except at the vertices of ABC. It is a curve with three superlinear branches of order f, and includes the case ot the three-cusped hypocycloid.

The only other type with no inflexion except those on a side of the triangle ABC is ($n00$, $0n0$, $00n$), discussed in MESSENGER OF MATHEMATICS, vol. 49 (1920), p. 132. Another interesting type is ($fF0$, $F0f$, $0fF$), discussed in this BULLETIN, vol. 29 (1923), p. 303. We may also note the type ($fF0$, $Ff0$, $00n$) with two real inflexions if $f \geqq \frac{1}{2}(n+\sqrt{n})$, especially the case in which $f=n-1$. The number of sextactic points not on a side of the triangle ABC is twelve or less.

If a node of (6) has parameters t_1 and t_2 (not 0, 1, or ∞) and

$$t_1/t_2 = \omega, \quad (t_1-1)/(t_2-1) = \Omega, \tag{11}$$

we have ω and $\Omega \neq 1$, $\omega \neq \Omega$, while

$$\omega^a \Omega^b = 1, \ \omega^c \Omega^d = 1. \tag{12}$$

These give

$$\omega^\varepsilon = 1, \ \Omega^\varepsilon = 1, \text{ where } \varepsilon = ad-bc, \tag{13}$$

whence

$$t_1 = \omega(1-\Omega)/(\omega-\Omega), \ t_2 = (1-\Omega)/(\omega-\Omega). \tag{14}$$

It is at once seen that the values of t_1 and t_2 given by (12), (13), (14) are complex and that the corresponding values of x and y given by (6) are real. Hence we have the following result.

The only point-singularities of the curve other than those on the sides of the triangle ABC are real acnodes.

If we take

$$\omega = e^{2h\pi i/\varepsilon}, \ \Omega = e^{2k\pi i/\varepsilon}, \tag{15}$$

the equation (12) becomes

$$ha+kb \equiv 0 \pmod{\varepsilon}, \ hc+kd \equiv 0 \pmod{\varepsilon} \tag{16}$$

and the acnodes are given on finding values of h and k between 1 and $\varepsilon-1$ satisfying (16). We may simplify (16) by putting

$$a=\lambda a_1, \ c=\lambda c_1, \ b=\mu b_1, \ d=\mu d_1, \ \varepsilon=\lambda\mu\varepsilon_1=\lambda\mu(a_1d_1-b_1c_1),$$

where a_1 is prime to c_1 and b_1 to d_1. We readily find

from (16), on eliminating h or k, that $h = \mu h_1$, $k = \lambda k_1$, where h_1 and k_1 are integral, and we get

(17) $h_1 a_1 + k_1 b_1 \equiv 0 \pmod{\varepsilon_1}$, $h_1 c_1 + k_1 d_1 \equiv 0 \pmod{\varepsilon_1}$.

The nature of the singularities on the sides of the triangle ABC is readily determined. For instance, suppose in (6) $c > a > 0$. Then (6) gives an expansion for t in ascending powers of $x^{1/a}$, and thence we get for y an expansion of the form

$$y = x^{c/a}(\alpha + \beta x^{1/a} + \gamma x^{2/a} + \cdots)$$

in general, fixing the nature of the singularity for which t is zero.

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SURFACES WITH ORTHOGONAL LOCI OF THE CENTERS OF GEODESIC CURVATURE OF AN ORTHOGONAL SYSTEM*

BY MALCOLM FOSTER

We consider a surface S referred to any orthogonal system. Let G_1 and G_2 be the centers of geodesic curvature of the curves $u =$ const. and $v =$ const. respectively, through any point M of S. As M is displaced over the entire surface the loci of G_1 and G_2 will in general be two surfaces S_1 and S_2, corresponding elements of which are those which result from a common displacement of M. We ask: What are the surfaces S for which the surfaces S_1 and S_2 correspond with orthogonality of linear elements?

The condition that the displacements of G_1 and G_2 be orthogonal for every displacement of M, is that the absolute displacements of these points in the directions of the axes of the moving trihedral at M satisfy the relation

(1) $$\sum \delta x_1 \, \delta x_2 = 0,$$

* Presented to the Society, April 28, 1923.

for all values of dv/du. The x-axis of the trihedral is chosen tangent to the curve $v = \text{const.}$ The radii of geodesic curvature of the curves $u = \text{const.}$ and $v = \text{const.}$ we denote by ϱ_{gv} and ϱ_{gu} respectively. The relation (1) becomes*

$$\varrho_{gu} r_1 \left(-\frac{\partial \varrho_{gv}}{\partial u}du - \frac{\partial \varrho_{gv}}{\partial v}dv + \xi du\right)dv + \varrho_{gv} r \left(\frac{\partial \varrho_{gu}}{\partial u}du + \frac{\partial \varrho_{gu}}{\partial v}dv + \eta_1 dv\right)du - \varrho_{gv}\varrho_{gu}(pdu + p_1 dv)(qdu + q_1 dv) \equiv 0.$$

Hence, setting the coefficients of du^2, $dudv$, and dv^2 equal to zero, we get

$$(2)\quad \begin{cases} \varrho_{gv}\left(\varrho_{gu} pq - r\dfrac{\partial \varrho_{gu}}{du}\right) = 0, \\ \varrho_{gu} r_1 \left(\xi - \dfrac{\partial \varrho_{gv}}{\partial u}\right) + \varrho_{gv} r \left(\eta_1 + \dfrac{\partial \varrho_{gu}}{\partial v}\right) - \varrho_{gv}\varrho_{gu}(pq_1 + p_1 q) = 0, \\ \varrho_{gu}\left(\varrho_{gv} p_1 q_1 + r_1 \dfrac{\partial \varrho_{gv}}{\partial v}\right) = 0. \end{cases}$$

Now $\varrho_{gv} = \dfrac{\eta_1}{r_1}$, and $\varrho_{gu} = \dfrac{\xi}{r}$;† using these values, and the relations between the fundamental quantities for the surface‡, the equations (2) reduce to

$$(3)\quad \begin{cases} \varrho_{gu} pq - r\dfrac{\partial \varrho_{gu}}{\partial u} = 0, \\ pq_1 = 0, \\ \varrho_{gv} p_1 q_1 + r_1 \dfrac{\partial \varrho_{gv}}{\partial v} = 0, \end{cases}$$

since $\varrho_{gv}, \varrho_{gu} \neq 0$.

Since $p = D'/\eta$, and $q_1 = -D'/\xi$,§ we see from the second member of (3) that both p and q_1, are zero, and that the parametric curves must be the lines of curvature.

* Eisenhart, *Differential Geometry of Curves and Surfaces*, p. 170.
† Eisenhart, p. 132, formula (47), and p. 167, formulas (45).
‡ Eisenhart, p. 168 and p. 170.
§ Eisenhart, p. 174, formulas (73).

Consequently the first and third members of (3) reduce to*

$$\frac{\partial \varrho_{gu}}{\partial u} = 0,$$

$$\frac{\partial \varrho_{gv}}{\partial v} = 0.$$

Hence

$$\varrho_{gv} = U, \qquad \varrho_{gu} = V, \tag{4}$$

where U and V are functions of u and v alone respectively. The parametric curves therefore have constant geodesic curvature and the system is isothermal.† The surface is therefore isothermal.

Making use of (4), we see that the elements ds_1^2 and ds_2^2 of the loci of G_1 and G_2 respectively, are

$$\begin{cases} ds_1^2 = \sum \delta x_1^2 = \dfrac{\eta_1^2}{r_1^2}\left[\dfrac{\left(\dfrac{\partial r_1}{\partial u}\right)^2}{r_1^2} + r^2 + q^2\right] du^2, \\ ds_2^2 = \sum \delta x_2^2 = \dfrac{\xi^2}{r^2}\left[\dfrac{\left(\dfrac{\partial r}{\delta v}\right)^2}{r^2} + r_1^2 + p_1^2\right] dv^2. \end{cases} \tag{5}$$

Hence the loci of G_1 and G_2 are curves and not surfaces. As the vertex of the trihedral describes a curve $u = \text{const.}$ the point G_1 remains fixed, and as the vertex of the trihedral describes a curve $v = \text{const.}$ the point G_2 remains fixed. The lines of curvature are therefore spherical in both systems; they lie on spheres whose centers lie on the loci of G_1 and G_2, and which are mutually orthogonal with S at every point.

We denote the curves which are the loci of G_1 and G_2 by Γ_1 and Γ_2 respectively. The curve Γ_1 is described by G_1 as the vertex of the trihedral describes every curve $v = \text{const.}$, and Γ_2 is described by G_2 as the vertex of

* We exclude the cases where either $r = 0$, or $r_1 = 0$, since in either case the curves in one family are geodesics, and one of the points G_1 and G_2 is at infinity.

† Eisenhart, p. 137.

the trihedral describes every curve $u = \text{const.}$ Suppose that the vertex of the trihedral describes a definite curve $u = \text{const.}$ The point G_1 remains fixed at some point on Γ_1 while G_2 describes Γ_2, and every tangent to Γ_2 will be perpendicular to the fixed tangent to Γ_1 at G_1. The curve Γ_2 is therefore either a plane curve whose plane is perpendicular to the fixed tangent to Γ_1, or a straight line perpendicular to this fixed tangent. Suppose now that the vertex of the trihedral describe a second curve $u = \text{const.}$ Then G_1 remains fixed at some second point of Γ_1 while G_2 describes Γ_2, and every tangent to Γ_2 will be perpendicular to the fixed tangent to Γ_1 at the second position of G_1. Consequently if Γ_2 be a plane curve, the locus of G_1, namely Γ_1, must be a straight line. We obtain similar conclusions if we consider the vertex of the trihedral to describe two different curves $v = \text{const.}$ Hence the locus of at least one of the points G_1 and G_2 is a straight line.

We suppose that it is the locus of G_1 which is a straight line. Consider the absolute displacements of the point G_1 in the directions of the axes of the trihedral T_u of a curve $v = \text{const.}$ We have*

$$\frac{\delta x_1}{ds} = \frac{dx_1}{ds} + 1, \quad \frac{\delta y_1}{ds} = \frac{x_1}{\varrho}, \quad \frac{\delta z_1}{ds} = 0, \tag{6}$$

where ds is the element of arc of the curve $v = \text{const.}$, and ϱ is the radius of first curvature. From the third member of (6), we see that the line Γ_1 which is the locus of G_1, lies in the osculating plane of the curve $v = \text{const.}$ at every point. Now

$$x_1 = -\varrho_{gv} = -\frac{\eta_1}{r_1},\dagger \qquad ds = \sqrt{E}\,du = \xi du;$$

using these relations, together with the relations between the fundamental quantities for the surface,‡ equations (6) become

* Eisenhart, p. 32.

† It is necessary tha ρ_{gv} be measured in the opposite direction to that in which the parameter u increases. Cf. Darboux, vol. II, p. 359.

‡ Eisenhart, p. 168 and p. 170.

$$(7)\qquad \frac{\delta x_1}{ds}=\frac{\eta_1\dfrac{\partial r_1}{\partial u}}{\xi r_1^2},\quad \frac{\delta y_1}{ds}=-\frac{\eta_1}{\varrho r_1},\quad \frac{\delta z_1}{ds}=0.$$

Hence

$$(8)\qquad ds_1=\frac{\eta_1\left[\varrho^2\left(\dfrac{\partial r_1}{\partial u}\right)^2+\xi^2 r_1^2\right]^{1/2}ds}{\varrho\xi r_1^2},$$

where ds_1 is the element of the line Γ_1. The direction-cosines of Γ_1 relative to the trihedral T_u are therefore

$$(9)\ \alpha_1=\frac{\varrho\dfrac{\partial r_1}{\partial u}}{\sqrt{\varrho^2\left(\dfrac{\partial r_1}{\partial u}\right)^2+\xi^2 r_1^2}},\quad \beta_1=\frac{-\xi r_1}{\sqrt{\varrho^2\left(\dfrac{\partial r_1}{\partial u}\right)^2+\xi^2 r_1^2}},\quad \gamma_1=0.$$

Since Γ_1 is a straight line fixed in space, we must have

$$\frac{\delta\alpha_1}{ds}=\frac{\delta\beta_1}{ds}=\frac{\delta\gamma_1}{ds}=0.$$

These equations become on using (9),*

$$(10)\quad \begin{cases} \dfrac{\delta\alpha_1}{ds}=\dfrac{d\alpha_1}{ds}-\dfrac{\beta_1}{\varrho} \\ =\dfrac{r_1^2\left[\varrho\xi\left(\varrho\dfrac{\partial^2 r_1}{\partial u^2}+\dfrac{\partial\varrho}{\partial u}\dfrac{\partial r_1}{\partial u}\right)-\varrho^2\dfrac{\partial r_1}{\partial u}\dfrac{\partial\xi}{\partial u}+\xi^3 r_1\right]}{\varrho\left[\varrho^2\left(\dfrac{\partial r_1}{\partial u}\right)^2+\xi^2 r_1^2\right]^{3/2}}=0, \\ \dfrac{\delta\beta_1}{ds}=\dfrac{d\beta_1}{ds}+\dfrac{\alpha_1}{\varrho} \\ =\dfrac{r_1\dfrac{\partial r_1}{\partial u}\left[\varrho\xi\left(\varrho\dfrac{\partial^2 r_1}{\partial u^2}+\dfrac{\partial\varrho}{\partial u}\dfrac{\partial r_1}{\partial u}\right)-\varrho^2\dfrac{\partial r_1}{\partial u}\dfrac{\partial\xi}{\partial u}+\xi^3 r_1\right]}{\xi\left[\varrho^2\left(\dfrac{\partial r_1}{\partial u}\right)^2+\xi^2 r_1^2\right]^{3/2}}=0, \\ \dfrac{\delta\gamma_1}{ds}=-\dfrac{\beta_1}{\tau}=\dfrac{\xi\eta_1 r_1}{\tau\left[\varrho^2\left(\dfrac{\partial r_1}{\partial u}\right)^2+\xi^2 r_1^2\right]^{1/2}}=0, \end{cases}$$

* Eisenhart, p. 32.

where τ is the radius of second curvature of the curve $v = \text{const.}$ Hence we must have

$$(11)\qquad \begin{cases} \varrho\xi\left(\varrho\dfrac{\partial^2 r_1}{\partial u^2}+\dfrac{\partial\varrho}{\partial u}\dfrac{\partial r_1}{\partial u}\right)-\varrho^2\dfrac{\partial r_1}{\partial u}\dfrac{\partial\xi}{\partial u}+\xi^3 r_1=0, \\ \dfrac{1}{\tau}=0. \end{cases}$$

The curves $v = \text{const.}$ are therefore plane; and since they are spherical, they are circles. Consequently $\varrho = \text{const.}$, and the first member of (11) reduces to

$$(12)\qquad \varrho^2\left(\xi\frac{\partial^2 r_1}{\partial u^2}-\frac{\partial r_1}{\partial u}\frac{\partial\xi}{\partial u}\right)+\xi^3 r_1=0.$$

The relation (12) may also be written in the form

$$(13)\qquad \varrho^2\frac{\partial}{\partial u}\left(\frac{\dfrac{\partial r_1}{\partial u}}{\xi}\right)+\frac{\partial\eta_1}{\partial u}=0.$$

The line Γ_1 therefore lies in the plane of every curve $v = \text{const.}$, and consequently the surfaces have plane lines of curvature in one system for which all the planes pass through the straight line Γ_1; such surfaces are called surfaces of Joachimsthal.*

Finally, the surfaces considered are isothermal surfaces of Joachimsthal for which the lines of curvature which lie in coaxial planes are circles, and for which either the relation (12) holds, or the corresponding relation is satisfied with reference to the curves $u = \text{const.}$

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* Eisenhart, pp. 308–310.

QUADRATIC FIELDS IN WHICH FACTORIZATION IS ALWAYS UNIQUE*

BY L. E. DICKSON

1. *Definitions.* Let m be an integer, other than 0 and 1, such that m is not divisible by a perfect square exceeding unity. All numbers $r+s\sqrt{m}$ in which r and s are rational constitute a *field* $R(\sqrt{m})$. Its *algebraic integers* are known to be $x+y\theta$, where x and y are rational integers, and

(1) $\theta=\sqrt{m}$ if $m\equiv 2$ or $m\equiv 3 \pmod 4$,

(2) $\theta=\frac{1}{2}(1+\sqrt{m})$, $\theta^2=\theta-k$, if $m\equiv 1 \pmod 4$,

where $k=\frac{1}{4}(1-m)$. The *conjugate* of $\xi=x+y\theta$ is defined to be $\xi'=x+y\theta'$, where $\theta'=-\theta$ in case (1), and $\theta'=\frac{1}{2}(1-\sqrt{m})$ in case (2). The product $\xi\xi'$ is called the *norm* of ξ, and is denoted by $N(\xi)$. According as the case is (1) or (2), we have

(3) $N(x+y\theta)=x^2-my^2$ or $x^2+xy+ky^2$.

If ξ is an algebraic integer such that $N(\xi)=\pm 1$, then ξ is called a *unit*. The only units in $R(i)$ are ± 1 and $\pm i$.

2. *Object of the Paper.* It is known† that -1, -2, -3, -7 and -11 are the only negative values of m for which the greatest common divisor process yielding numerically decreasing norms is always applicable in $R(\sqrt{m})$, so that if a and b are any algebraic integers ($b\neq 0$) there exist algebraic integers q and r of the field such that

$$a=bq+r, \quad |\text{norm } r|<|\text{norm } b|.$$

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* Presented to the Society, December 29, 1923. See also This BULLETIN, p. 90, Jan.–Feb., 1924, and footnote, p. 247, May-June, 1924.

† For a geometric proof, see Birkhoff, AMERICAN MATHEMATICAL MONTHLY, vol. 13 (1906), pp. 156–159.

For a few positive values of m, as 2, 3, 5, 11, such a process exists. But there are many values of m for which this process is not applicable, although there exists a greatest common divisor as shown by the theory of ideals when the number of classes of ideals is unity.

Avoiding the theory of ideals, we shall give an elementary proof of the following result.

THEOREM 1. *Let m be any integer for which there is a single* class of properly primitive binary quadratic forms capable of representing positive integers and having the discriminant $4m$ or m according as the case is* (1) *or* (2). *Then the algebraic integers of $R(\sqrt{m})$ admit unique factorization into primes apart from the association of unit factors.*†

This proof places at the disposal of students of elementary theory of numbers an effective tool utilized by Gauss and Dirichlet in the case $m = -1$. Moreover, the proof furnishes a model for the investigation‡ of the arithmetics of linear algebras for which no theory of ideals is available.

3. LEMMA 1. *If a and b are relatively prime and*

$$(4) \qquad a^2 - mb^2 = pq \quad \text{or} \quad a^2 + ab + kb^2 = pq, \quad p > 0,$$

there exist integers z and w such that

$$(5) \qquad p = z^2 - mw^2 \quad \text{or} \quad p = z^2 + zw + kw^2,$$

according as case (1) *or* (2) *holds.*

For, b and q are relatively prime since otherwise a common prime divisor of them would divide a^2 and hence a. Hence

* In this BULLETIN, vol. 17 (1910–11), pp. 534–37, the writer proved that there is a single class of positive primitive quadratic forms of negative discriminant $-P$ with $P < 1{,}500{,}000$ only when $P = 3, 4, 7, 8, 11, 12, 16, 19, 27, 28, 43, 67, 163$. But the cases in which the discriminant is positive are very numerous.

† Theorem 1 holds also if there are only two classes and these are *opposite* classes of properly primitive forms of discriminant $4m$ or m. In the first case, every such form is equivalent to $x^2 - my^2$ or $mx^2 - y^2$. The proof differs from that in the text only by the occasional insertion of the double sign $\pm$.

‡ See L. E. DICKSON, AMERICAN JOURNAL, 1924.

there exist integers s and t such that $a = sb + tq$. Inserting this in (4), we get

$$pq = Ab^2 + Bbqt + q^2t^2,$$

where $A = s^2 - m$, $B = 2s$ in case (1); while $A = s^2 + s + k$, $B = 2s + 1$ in case (2). Hence A must be divisible by q. Write $A = qe$. Then

$$p = eb^2 + Bbt + qt^2. \tag{6}$$

The discriminant of this form is $B^2 - 4eq = 4m$ or m according as the case is (1) or (2). If any odd prime divides e, B and q, its square divides $B^2 - 4eq$, whereas m has no square factor. If, in case (1), 2 divides both e and q, then $m = s^2 - eq \equiv s^2 \equiv 0$ or $1 \pmod 4$, contrary to (1). Hence in every case (6) is a properly primitive form representing the positive integer p and therefore, by our hypothesis of a single class, is equivalent to the respective form (5) of discriminant $4m$ or m.

4. LEMMA 2.* *If a and b are relatively prime integers and if $N(a + b\theta)$ is divisible by the rational prime p, then p decomposes and one of its integral algebraic factors divides $a + b\theta$.*

By hypothesis, we have (4) and hence (5).

(i) Let the first equations (4) and (5) hold. Then

$$mb^2 \equiv a^2, \quad z^2 \equiv mw^2 \pmod p.$$

By multiplication we get $bz \equiv \pm aw \pmod p$ if m is not divisible by p, and we may choose the upper sign after changing the sign of w if necessary. If m is divisible by p, we have $z \equiv 0$, $a \equiv 0 \pmod p$. Hence $bz \equiv aw \pmod p$ in all cases. Define rational numbers x and y by means of

$$p(x + y\theta) = (a + b\theta)(z - w\theta), \tag{7}$$

* Another proof follows from the writer's theorem in this BULLETIN, vol. 29 (1923), pp. 464–467, that all solutions of $N(a + b\theta) = pq$ are products of the same integer ρ by the numbers obtained from (8) and $p = N(z + w\theta)$, $q = N(x + y\theta)$. Here $\rho = \pm 1$ since a and b are relatively prime. But if we omit §§ 3-4, we must replace § 5 by one of the standard proofs of unique representation of a prime as a norm.

whence

$$x = \frac{az - mbw}{p}, \quad y = \frac{bz - aw}{p},$$

so that y is an integer. Since $p = (z + w\theta)(z - w\theta)$, we may cancel $z - w\theta$ from (7) and get

$$(z + w\theta)(x + y\theta) = a + b\theta. \tag{8}$$

Taking norms, we have $p(x^2 - my^2) = pq$, whence x^2 is the integer $my^2 + q$. Hence both x and y are integers in (8), which is the desired result.

(ii) Let the second equations (4) and (5) hold. Then

$$(2a + b)^2 - mb^2 = 4pq, \quad 4p = (2z + w)^2 - mw^2,$$

$$mb^2(2z + w)^2 \equiv (2a + b)^2 mw^2 \pmod{p}.$$

If m is not divisible by p, we get

$$b(2z + w) \equiv \pm(2a + b)w \pmod{p}. \tag{9}$$

First, let the upper sign hold in (9). Then, if $p \neq 2$, $bz \equiv aw \pmod{p}$. This follows also if m is divisible by p, whence $2a + b \equiv 0$, $2z + w \equiv 0$. Define rational numbers x and y by means of

$$p(x + y\theta) = (a + b\theta)(z + w\theta'), \tag{10}$$

whence

$$x = \frac{a(z + w) + kbw}{p}, \quad y = \frac{bz - aw}{p},$$

so that y is an integer. Since $p = (z + w\theta)(z + w\theta')$, we may cancel $z + w\theta'$ from (10) and get (8). By the norm of (8), $p(x^2 + xy + ky^2) = pq$. Thus x is a rational root of an equation with integral coefficients and leading coefficient unity; hence x is an integer.

Second, let the lower sign hold in (9). Then if $p \neq 2$, $b(z + w) \equiv -aw \pmod{p}$. Introduce the integers $Z = z + w$, $W = -w$. Then $bZ \equiv aW$ and

$$Z^2 + ZW + kW^2 = z^2 + zw + kw^2 = p, \tag{11}$$

and we are led to our first case with Z and W in place of z and w. Hence $a + b\theta$ has the factor $Z + W\theta = z + w\theta'$.

Finally, let $p = 2$. If b is even, so that a is odd, $a^2 + ab$

$+kb^2$ is odd, whereas it is divisible by $p=2$ by (4). If w is even, z is even in $2=z^2+zw+kw^2$, which is then divisible by 4. Hence b and w are both odd. By (11) we may add w to z and hence make z even or odd at pleasure and hence make $z\equiv a \pmod 2$. Then $bz\equiv aw \pmod 2$. We proceed as in the first case.

5. LEMMA 3. *If a rational prime p is expressible as a norm, it is the product of two algebraic integers, neither a unit, in one and only one way apart from unit factors and apart from the arrangement of the two integers.*

We have $p=(z+w\theta)(z+w\theta')$. Let also $p=\pi\varrho$, where $\pi=a+b\theta$ and ϱ are algebraic integers neither a unit. Then $p^2=N(\pi)N(\varrho)$, whence $N(\pi)=N(\varrho)=\pm p$. If a and b had a common prime factor, its square would divide $N(\pi)=\pm p$. Hence we may apply the proof of Lemma 2 with $q=\pm 1$ and obtain (8) or the similar equation with $z+w\theta$ replaced by $z+w\theta'$. Then $x+y\theta$ is a unit since its norm is $q=\pm 1$. Hence every factorization $p=\pi\varrho$ differs from the given one only by the insertion of unit factors or by the interchange of the given factors.

6. *Algebraic Primes.* An algebraic integer not a unit of $R(\theta)$ is called an *algebraic prime* if it is not a product of two algebraic integers neither a unit of $R(\theta)$.

If a rational prime p is a norm, so that $p=\pi\pi'$, then π and π' are algebraic primes. For, if $\pi=\alpha\beta$, where neither α nor β is a unit, then $p=N(\alpha)N(\beta)$, and one of the norms is ± 1, so that α or β is a unit.

7. LEMMA 4. *If $N(c+d\theta)$ is divisible by a rational prime p, then either $c+d\theta$ is divisible by p or else p decomposes and $c+d\theta$ is divisible by one of the algebraic prime factors of p.*

Let g be the greatest common divisor of $c=ga$, $d=gb$. Then

$$c+d\theta=g(a+b\theta) \tag{12}$$

is divisible by p if g is. Next, let g be not divisible by p. Since $N(c+d\theta) = q^2N(a+b\theta)$ is divisible by p, $N(a+b\theta)$ is divisible by p and Lemma 2 shows that p decomposes and that $a+b\theta$ is divisible by one of the algebraic prime factors of p. Hence (12) is divisible by that factor.

8. THEOREM 2. *If an algebraic prime divides a product AB, it divides A or B.*

(i) Suppose the algebraic prime is a rational prime p. We may write $A = q\alpha$, where q is a rational integer and α has relatively prime coordinates. Then p divides qP, where $P = \alpha B = r+s\theta$. Then p divides qr and qs. If p divides q, it divides A. In the contrary case, p divides r, s and hence also αB. Write $B = t\beta$, where β has relatively prime coordinates. As before, either p divides t and hence B, or else p divides $\alpha\beta$, so that p^2 divides $N(\alpha)N(\beta)$. We may then assume that p divides $N(\alpha)$ for example, whence, by Lemma 2, p decomposes, whereas it is an algebraic prime.

(ii) Let π be an algebraic prime not the product of an integer by a unit. Hence the coordinates of π are relatively prime. The integer $N(\pi)$ is divisible by a rational prime p. Hence, by Lemma 2, $p = \varrho\varrho'$ and ϱ is a divisor of π. Thus the prime π is the product of ϱ by a unit, whence $N(\pi) = \pm N(\varrho) = \pm p$. Suppose that π divides AB, but divides neither A nor B. Then $\pm p = N(\pi)$ divides $N(A)N(B)$. Let therefore p divide $N(A)$ for example. By Lemmas 3 and 4, A is divisible by π or π'. Hence A is divisible by π'. Write $A = \pi'\alpha$, $P = \alpha B$. Then π divides $AB = \pi'P$, so that $p = \pm\pi\pi'$ divides π'^2P. We shall prove that p divides $\pi'P$. Write

$$\pi' = a+b\theta,\ P = c+d\theta,\ \pi'^2P = r+s\theta,$$
$$r \equiv s \equiv 0 \pmod{p}.$$

First, let $\theta = \sqrt{m}$. Then $\pm p = a^2 - mb^2$ and

$$r = a^2c+2mabd+mb^2c, \quad s = a^2d+2abc+mb^2d.$$

Elimination of a^2 by means of $a^2 \equiv mb^2 \pmod{p}$ gives

$$r \equiv 2mb(ad+bc) \equiv 0, \quad s \equiv 2b(ac+mbd) \equiv 0 \pmod{p}.$$

If $2mb$ is not divisible by p, the numbers in parenthesis, which are the coordinates of $\pi' P$, are divisible by p. Hence P is divisible by π. Thus π divides αB, but divides neither α nor B. A repetition of the argument leads evidently to a contradiction. Next, if b were divisible by p, a would be also in view of $\pm p = a^2 - mb^2$, which would then be divisible by p^2. If m were divisible by p, a would be also, whence π'^2 is divisible by p, and hence π' by π, so that π' is the product of π by a unit. Since A is divisible by π', it is divisible by π. If $p = 2$, $\pi'^2 \equiv a^2 + b^2 m \equiv 0 \pmod{p}$.

Second, consider case (2). Then $\pm p = a^2 + ab + kb^2$, so that b is not divisible by p. In

$$r = a^2 c - 2kabd - kb^2 c - kb^2 d,$$
$$s = a^2 d + b^2 c + 2abc + 2abd + (1-k) b^2 d$$

we replace a^2 by $-ab - kb^2 \pmod{p}$, cancel factors b, and get

$$\begin{aligned} 2kad + ac + 2kbc + kbd &\equiv 0, \\ bc + 2ac + ad + (1-2k)bd &\equiv 0 \pmod{p}. \end{aligned} \tag{13}$$

Elimination of ac gives $(4k-1)E \equiv 0$, where $E = ad + bc + bd$. But if $4k - 1 = -m$ were divisible by p, $\pm 4p = (2a+b)^2 - mb^2$ shows that $2a + b \equiv 0$, whence

$$\pi'^2 = a^2 - kb^2 + b(b+2a)\theta \equiv a^2(1-4k) = ma^2 \equiv 0 \pmod{p},$$

so that A is divisible by π as in the first case. In view of this contradiction, we have $E \equiv 0$. Adding $-2kE$ to the first equation (13), we get $F \equiv ac - kbd \equiv 0$. Hence

$$\pi' P = F + E\theta \equiv 0 \pmod{p}.$$

THE UNIVERSITY OF CHICAGO

THE JACOBIAN OF A CONTACT TRANSFORMATION*

BY E. F. ALLEN

The equations

$$(1)\quad x_1 = X(x, z, p),\quad z_1 = Z(x, z, p),\quad p_1 = P(x, z, p),$$

where X, Z, and P are functions of class C″, represent a transformation of line-elements in the xz plane to line-elements in the x_1z_1 plane. With Lie we shall define every transformation in x, z, p, which leaves the Pfaff differential equation

$$(2)\quad \cdot dz - p\,dx = 0$$

invariant, as a contact transformation of the xz plane to the x_1z_1 plane. Hence the equations (1) must satisfy an identity of the form

$$(3)\quad dz_1 - p_1\,dx_1 = \varrho(dz - p\,dx),$$

where ϱ is a function of x, z, and p alone.

The following relations connecting X, Z, P, and their partial derivatives are easily obtained:†

$$(4)\quad \begin{cases} Z_x - PX_x = -p\varrho, \\ Z_z - PX_z = \varrho, \\ Z_p - PX_p = 0; \end{cases}$$

$$(5)\quad \begin{cases} [XZ] = X_p(Z_x + pZ_z) - Z_p(X_x + pX_z) = 0, \\ [PX] = \varrho, \text{ and } [PZ] = \varrho P. \end{cases}$$

The jacobian of transformation (1) is

$$(6)\quad J = \begin{vmatrix} X_x & X_z & X_p \\ Z_x & Z_z & Z_p \\ P_x & P_z & P_p \end{vmatrix}.$$

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* Presented to the Society, December 1, 1923.

† Lie und Scheffers, *Geometrie der Berührungstransformationen*, p. 68, Chap. 3.

We shall show that this jacobian is equal to ϱ^2. Let us multiply the first row by P and subtract the product from the second row; then

$$J = \begin{vmatrix} X_x & X_z & X_p \\ Z_x - PX_x & Z_z - PX_z & Z_p - PX_p \\ P_x & P_z & P_p \end{vmatrix}. \tag{7}$$

Hence, using equations (4), we find

$$J = \begin{vmatrix} X_x & X_z & X_p \\ -p\varrho & \varrho & 0 \\ P_x & P_z & P_p \end{vmatrix}. \tag{8}$$

This reduces to

$$J = \begin{vmatrix} X_x + pX_z & X_z & X_p \\ 0 & \varrho & 0 \\ P_x + pP_z & P_z & P_p \end{vmatrix}, \tag{9}$$

when the second column is multiplied by p and the sum is added to the first column. Evaluating this determinant, we have

$$J = \varrho[P_p(X_x + pX_z) - X_p(P_x + pP_z)]. \tag{10}$$

Therefore, by equation (5), we may write

$$J = \varrho^2. \tag{11}$$

The equations of a contact transformation may be regarded as the equations of a point transformation, which transforms points in xzp space to points in $x_1z_1p_1$ space. In general a surface in xzp space, represented by the equation $F_1(x, z, p) = 0$, will be transformed into a surface in $x_1z_1p_1$ space, represented by the equation $F_2(x_1, z_1, p_1) = 0$. Or if we regard equations (1) as the equations of a transformation of line-elements, it will transform a differential equation in x, z, p, into one in x_1, z_1, p_1, and also the solutions of the first differential equation into the solutions of the second.

Now if we set ϱ equal to zero,* we will have the equa-

* In some cases there are no values of the variables that will make ρ equal to zero. The following theory does not apply to such cases.

tion of a surface in xzp space, or we might say that we have a differential equation in xz space. Let us see into what this surface or into what this differential equation is transformed when it is subjected to the transformation (1). A few examples result in obtaining curves in $x_1z_1p_1$ space or in obtaining differential equations free from p_1. This leads to the following theorem.

THEOREM. *The surface* $\varrho = 0$ *is transformed into a curve in space by the transformation* (1).

If the partial derivative of ϱ with respect to z is not identically equal to zero, the equation $\varrho = 0$ may be solved for z.* Assuming that this is true, when the value thus obtained for z is substituted in X, Z, and P, they become functions of x and p alone. Regarding x and p as parameters, the equations (1) are the parametric equations of a surface. A necessary and sufficient condition† that

$$x_1 = f(x, y), \quad y_1 = g(x, y), \quad z_1 = h(x, y) \tag{12}$$

define a curve in space and not a surface is that

$$EG - F^2 = A^2 + B^2 + C^2 \equiv 0, \tag{13}$$

where

$$A = \frac{\partial(y_1, z_1)}{\partial(x, y)}, \quad B = \frac{\partial(z_1, x_1)}{\partial(x, y)}, \quad C = \frac{\partial(x_1, y_1)}{\partial(x, y)}. \tag{14}$$

To prove our theorem it is necessary and sufficient to show that the A, B, and C connected with equations (1) are identically equal to zero. That is, it is sufficient to show that all the determinants of the following matrix vanish identically:

$$\begin{vmatrix} X_x + pX_z & X_p + \dfrac{\partial z}{\partial p} X_z \\ Z_x + pZ_z & Z_p + \dfrac{\partial z}{\partial p} Z_z \\ P_x + pP_z & P_p + \dfrac{\partial z}{\partial p} P_z \end{vmatrix}. \tag{15}$$

* If $\partial\rho/\partial z \equiv 0$ we will be able to solve for either x or p if $\rho \neq$ const.

† Eisenhart, *Differential Geometry*, p. 71.

Let us see what the effect will be when the value of z as obtained from $\varrho = 0$ is substituted in equations (1). Suppose that the substitution has been made in X and Z. It is easy to see that X_z and Z_z are equal to zero, and that to differentiate X completely with respect to x, it is necessary to differentiate with respect to x and then to use the function of a function rule, thus $X_x + X_z(\partial z/\partial x)$, and similarly for the other letters. Thus using the fact that $\varrho = 0$, we may write the equations (5) in the form

$$(16)\begin{cases} \left(X_p+\dfrac{\partial z}{\partial p}X_z\right)(Z_x+pZ_z)-\left(Z_p+\dfrac{\partial z}{\partial p}Z_z\right)(X_x+pX_z)=0, \\ \left(P_p+\dfrac{\partial z}{\partial p}P_z\right)(X_x+pX_z)-\left(X_p+\dfrac{\partial z}{\partial p}X_z\right)(P_x+pP_z)=0, \\ \left(P_p+\dfrac{\partial z}{\partial p}P_z\right)(Z_x+pZ_z)-\left(Z_p+\dfrac{\partial z}{\partial p}Z_z\right)(P_x+pP_z)=0. \end{cases}$$

It is very easy to see that these equations are now the expanded form of the determinants of the matrix (15). Hence the theorem is proved.

THE UNIVERSITY OF MISSOURI

INTEGRO-DIFFERENTIAL INVARIANTS OF ONE-PARAMETER GROUPS OF FREDHOLM TRANSFORMATIONS*

BY A. D. MICHAL

1. *Statement of the Problem.* The author† has already considered functionals of the form $f[y(\tau_0'), y'(\tau_0')]$ (depending only on a function $y(\tau)$ and its derivative $y'(\tau)$ between 0 and 1) which are invariant under an arbitrary Volterra one-parameter group of continuous transformations. The

* Presented to the Society, December 1, 1923.

† Cf. *Integro-differential expressions invariant under Volterra's group of transformations* in a forthcoming issue of the ANNALS OF MATHEMATICS. This paper will be referred to as "*I. D. I. V.*"

calculation of the invariants in question was effected in the case of a large class of functionals known as analytic functionals.

The purpose of this note is to consider the problem of finding analytic functionals $f[y(\tau_0'), y'(\tau_0')]$ invariant under a Fredholm group of transformations

$$y_1(x) = y(x) + \int_0^1 K(x, s \mid a) y(s) ds, \tag{1}$$

where a is the parameter of this continuous one-parameter group of transformations, and where $a = 0$ corresponds to the identical transformation.

We restrict ourselves to transformations (1) for which the $y'(\tau)$'s exist and are continuous in the interval $I: 0 \leqq \tau \leqq 1$; $K(x, s \mid a)$ and $\partial K/\partial x$ are continuous in x and s in the square $S: 0 \leqq x \leqq 1$, $0 \leqq s \leqq 1$; and $\partial K/\partial x$ is not identically zero when $a \neq 0$.

The infinitesimal transformation corresponding to (1) will be of the form

$$\delta y(x) = \left[\int_0^1 H(x, s) y(s) ds\right] \delta a \tag{2}$$

with

$$\delta y'(x) = \left[\int_0^1 H_1(x, s) y(s) ds\right] \delta a, \tag{3}$$

where

$$H_1(x, s) \equiv \frac{\partial H(x, s)}{\partial x},$$

as the extended group of infinitesimal transformations.

Here follow the well known relations* between the kernel $K(x, s \mid a)$ of the Fredholm finite transformation (1) and the kernel $H(x, s)$ of the corresponding infinitesimal transformation (2):

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* Gerhard Kowalewski, *Über Funktionenräume*, WIENER SITZUNGSBERICHTE, 1911, vol. 120, IIA.

$$K(x, s \mid a) = \sum_{i=1}^{\infty} \frac{a^i H^i}{i!}, \tag{4}$$

$$H(x, s) = \left[\frac{\partial K(x, s \mid a)}{\partial a}\right]_{a=0} = \frac{1}{a} \sum_{i=1}^{\infty} (-1)^{i-1} \frac{K^i}{i}, \quad a \neq 0, \tag{5}$$

where H^i and K^i are to be interpreted according to Volterra's symbolic multiplication.

By methods similar to those employed in proving the lemma of Part I of *I. D. I. V.*, we can prove* without difficulty the following lemma.

LEMMA. *Necessary and sufficient conditions that $H(x, s)$ and $\partial H/\partial x$ be continuous in x and s and that $\partial H/\partial x$ be not identically zero are that $K(x, s \mid a)$ and $\partial K/\partial x$ be continuous in x and s and that $\partial K/\partial x$ be not identically zero, when $a \neq 0$.*

2. *A Sufficient Condition for Invariance.*

THEOREM 1. *Let $f[y(\tau_0'), y'(\tau_0')]$ be an analytic functional of $y(\tau)$ and $y'(\tau)$,* i. e., *developable in a Volterra expansion*†

$$f_{00} + \sum_{j=1}^{\infty} \left\{ \frac{1}{j!} \int_0^1 \int_{0_{(j)}}^1 \cdots \int_0^1 \left[\sum_{k=0}^{j} \binom{j}{k} f_{j-k,k}(t_1, \cdots, t_{j-k}; t_{j-k+1}, \cdots, t_j) \times \prod_{i=1}^{j-k} y(t_i) \prod_{i=j-k+1}^{j} y'(t_i) \right] dt_1 dt_2 \cdots dt_j \right\}. \tag{6}$$

We shall assume that $f_{j-k,k}$ is continuous in its j arguments, symmetric separately in the sets of arguments $t_1, t_2, \ldots, t_{j-k}$ and $t_{j-k+1}, \ldots, t_j$ respectively; and for convenience we assume also that

$$|f_{j-k,k}| < \gamma, \qquad |y| < \varrho_1, \qquad |y'| < \varrho_2, \tag{7}$$

where γ, ϱ_1, ϱ_2 are positive constants. Then a sufficient condition that $f[y(\tau_0'), y'(\tau^0)]$ be invariant under a given group of transformations (1) *is that it satisfy the relation*

* The proof comes by a direct calculation of the series involved.

† This is a generalization of Taylor's series given by Volterra. See for example his *Leçons sur les Équations Intégrales*, 1913.

$$(8)\qquad \int_0^1 H_1(t, t_{i+k}) f_{i,k}(t_1, \ldots, t_i; t_{i+1}, \ldots, t_{i+k-1}, t)dt$$
$$-\equiv \int_0^1 H(t, t_{i+k}) f_{i+1,k-1}(t, t_1, \ldots, t_i; t_{i+1}, \ldots, t_{i+k-1})dt.$$

The necessary and sufficient condition that $f[y(\tau_0'), y'(\tau_0)]$ be invariant under (1) is that under (2)

$$(9)\qquad \delta f[y(\tau_0'), y'(\tau_0')] \equiv 0 \text{ in } y \text{ and } y'.$$

Since the analyticity of our functionals insures the validity of a Volterra variation, we may use Volterra's* form of the variation of a functional. Then condition (9) becomes

$$(10)\qquad \int_0^1 f_y(t)\delta y(t)dt + \int_0^1 f_{y'}(t)\delta y'(t)dt \equiv 0$$

in y and y', where $f_y(t)$ and $f_{y'}(t)$ are the partial functional derivatives of $f[y(\tau_0'), y'(\tau_0')]$ with respect to $y(\tau)$ and $y'(\tau)$, respectively, both taken at the point t.

Substituting in (10) the values of $\delta y(t)$ and $\delta y'(t)$ as given by (2) and (3), respectively, rearranging and dividing through by δa, we get

$$(11)\qquad \int_0^1 y(s)\Big[\int_0^1 f_y(t)H(t, s)dt + \int_0^1 f_{y'}(t)H_1(t, s)dt\Big]ds \equiv 0$$

in y. We may now apply Lemma 2 of *I. D. I. V.*; doing so, we find

$$(12)\qquad \int_0^1 f_{y'}(t)H_1(t, s)dt = -\int_0^1 f_y(t)H(t, s)dt.$$

Such operations as functional differentiations term by term are valid since the series involved are uniformly convergent under our hypotheses.† Calculating the partial functional derivatives $f_y(t)$ and $f_{y'}(t)$, respectively, and substituting them in (12), we get by an easy reduction‡

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* A more general expression for δf would be in the form of Stieltjes integrals.

† Cf. *I. D. I. V.*, Part II, and Volterra's *Leçons sur les Équations Intégrales*, 1913, p. 18.

‡ Cf. similar reductions of *I. D. I. V.*

$$\sum_{j=1}^{\infty}\Big\{\frac{1}{(j-1)!}\int_0^1\int_{0_{(j)}}^1\cdots\int_0^1\Big[\sum_{l=0}^{j-1}\binom{j-1}{l}H_1(t,s)f_{j-1-l,\,l+1}(t_1,\ldots$$

$$\ldots,t_{j-1-l};\,t_{j-l},\ldots,t_{j-1},t)$$

$$\times\prod_{i=1}^{j-1-l}y(t_i)\prod_{i=j-l}^{j-1}y'(t_i)\Big]dt_1\ldots dt_{j-1}dt\Big\}$$

(13)

$$\equiv-\sum_{j=1}^{\infty}\Big\{\frac{1}{(j-1)!}\int_0^1\int_{0_{(j)}}^1\cdots\int_0^1\Big[\sum_{k=0}^{j-1}\binom{j-1}{k}H(t,s)f_{j-k,k}(t,t_1,\ldots$$

$$\ldots,t_{j-k-1};\,t_{j-k},\ldots,t_{j-1})$$

$$\times\prod_{i=1}^{j-k-1}y(t_i)\prod_{i=j-k}^{j-1}y'(t_i)\Big]dt_1\ldots dt_{j-1}dt\Big\},$$

where $k=l+1$. Equating coefficients of similar terms in y and y', we find

$$\int_0^1 H_1(t,s)f_{j-1-l,\,l+1}(t_1,\ldots,t_{j-1-l};\,t_{j-l},\ldots,t_{j-1},t)dt$$

$$\equiv-\int_0^1 H(t,s)f_{j-l,\,l}(t,t_1,\ldots,t_{j-1-l};\,t_{j-l},\ldots,t_{j-1})dt,$$

which can be written in the form (8).

3. *Calculation of the Invariants* $f[y(\tau_0'),\,y'(\tau_0')]$. In order that $f[y(\tau_0'),\,y'(\tau_0')]$ be invariant under (1) it is sufficient that the following recurrence formula hold

$$f_{i,k}(t_1,\ldots,t_i;\,t_{i+1},\ldots,t_{i+k-1},t)$$

(14)

$$\equiv-\frac{H(t,t_{i+k})}{H_1(t,t_{i+k})}f_{i+1,\,k-1}(t,t_1,\ldots,t_i;\,t_{i+1},\ldots,t_{i+k-1}).$$

We shall now prove the following theorem.

THEOREM II. *A necessary and sufficient condition on* (1) *that an analytic functional* $f[y(\tau_0),\,y'(\tau_0')]$ *be invariant under* (1) *when* (14) *holds is that the kernel* $H(x,s)$ *of the infinitesimal transformation be of the form*

$$H(x,s)\equiv\psi(s)e^{x/c},\tag{15}$$

*where $\psi(s)$ is an arbitrary function of s, and where c is a constant.**

It is evident from (14) that

$$\frac{H(t, t_{i+k})}{H_1(t, t_{i+k})}$$

must be independent of t_{i+k}, and hence it is necessary that it be a function of t alone, say $\varphi(t)$. On applying (14) until $f_{i,k}$ is written in terms of f's with second index zero, we get the recurrence formula

$$(16) \qquad f_{i,k}(t_1, t_2, \ldots, t_i; t_{i+1}, \ldots, t_{i+k-1}, t) = (-1)^k[\varphi(t)]^k f_{i+k,0}(t, t_1, \ldots, t_i, t_{i+1}, \ldots, t_{i+k-1}).$$

By hypothesis $f_{i+k,0}$ is symmetric in all its arguments. Therefore, interchanging t_1 and t_{i+1} leaves the right-hand side of (16) unchanged. Hence if (16) is to hold, $f_{i,k}$ must be symmetric with respect to t_1 and t_{i+1}, and therefore it must be symmetric in all its arguments. On interchanging t and any t_j in (16), we see at once that $\varphi(t)$ must be a constant, say c; i. e., $H(x, s)$ must satisfy the equation

$$(17) \qquad c\frac{\partial H(x, s)}{\partial x} - H(x, s) = 0,$$

whose most general solution is (15).

We may now remark that the arbitrariness of the coefficients $f_{i+k,0}$, in terms of which all the other $f_{i,k}$'s can be evaluated, on making use of the recurrence formula, enables us to state immediately the following theorem.

THEOREM III. *Let the kernel $H(x, s)$ of the infinitesimal transformation (2) be of the form $\psi(s)e^{x/c}$, and let us take an analytic functional $f[y(\tau_0'), y'(\tau_0')]$, all of whose $f_{i,k}$'s are symmetric in all their arguments, and assign arbitrarily for initial conditions the coefficients $f_{i+k,0}$ in its Volterra expansion; that is, take an arbitrary $F[y(\tau_0')]$ such that $F[y(\tau_0')] \equiv f[y(\tau_0'), y_0'(\tau_0')]$, and for convenience take $y_0(\tau) \equiv 0$. Then, if the $f_{i,k}$'s are calculated by the recurrence formula*

* That is, if $H(x,s) \equiv \psi(s)e^{x/c}$, we may assert that invariant analytic functionals $f[y(\tau_0')\ y'(\tau_0')]$ always exist.

(18) $f_{i,k}(t_1, \ldots, t_{i+k}) = (-1)^k c^k f_{i+k,0}(t_1, \ldots, t_{i+k})$,

we shall have an analytic functional $f[y(\tau_0'), y'(\tau_0)]$ *which will be invariant under a transformation* (1) *whose kernel* $K(x, s \mid a)$ *is given by*

$$K(x, s \mid a)$$

$$(19) \quad = \sum_{i=1}^{\infty} \frac{a^i}{i!} \psi(s) e^{x/c} \int_0^1 \int_{0_{(i-1)}}^1 \cdots \int_0^1 e^{\frac{1}{c}\sum_{j=1}^{i-1} t_j} \prod_{l=1}^{i-1} \psi(t_l) dt_1 \ldots dt_{i-1}.$$

4. *Example.* We here give an easy example in which the direct verification by means of the finite transformation is very simple. Let us suppose that

$$\delta y(x) = \left[e^x \int_0^1 s y(s) ds\right] \delta a$$

is the given infinitesimal transformation, i. e., that $H(x, s) = se^x$. By means of an easy calculation, the finite transformation may be written in the form

$$(20) \qquad y_1(x) = y(x) + (e^a - 1) e^x \int_0^1 s y(s) ds,$$

i. e., $K(x, s \mid a) = (e^a - 1) e^x s$. Let us take for initial condition

$$f[y(\tau_0'), 0] \equiv F[y(\tau_0')]$$

$$\equiv f_{00} + \int_0^1 f_{10}(t_1) y(t_1) dt_1 + \frac{1}{2!} \int_0^1 dt_2 \int_0^1 f_{20}(t_1, t_2) y(t_1) y(t_2) dt_1.$$

Then the functional $f[y(\tau_0'), y'(\tau_0')]$ given by

$$f[y(\tau_0'), y'(\tau_0')] = f_{00} + \int_0^1 f_{10}(t_1)[y(t_1) - y'(t_1)] dt_1$$

$$+ \frac{1}{2!}\left[\int_0^1 dt_2 \int_0^1 f_{20}(t_1, t_2)\{y(t_1) y(t_2) - 2y(t_1) y'(t_2) + y'(t_1) y'(t_2)\}\right] dt$$

is invariant under (20).

THE RICE INSTITUTE

REDUCTIONS OF ENUMERATIONS IN HOMOGENEOUS FORMS*

BY E. T. BELL

1. *Introduction.* By carrying out the work in detail for the form $ax^2+by^2+cz^2$ we shall derive a useful set of reduction formulas, and illustrate a general process which can easily be applied to the reduction of the number $N(n=f)$ of representations of the integer n in any homogeneous form f of any degree in any number of variables. This set contains implicitly the complete set of corresponding reduction formulas for $Ax^2+By^2+Cz^2+\cdots+Et^2$, in any number of indeterminates $x, y, z, \ldots, t$. The formulas in no case yield by themselves a complete evaluation of $N(n=f)$ for any type of n, but in many instances they materially simplify the problem, either by making the evaluation for f depend upon that for a simpler form, or by reducing the n to be represented to a more tractable type. By means of the process developed here, combined with elliptic function expansions, I have recently obtained several new complete enumerations for special ternary and quinary quadratic forms; the results will be published in other papers.

Before proceeding to the main discussion it will be instructive to glance at what is known concerning $N(n=f)$ in the simplest case (other than f linear), viz., $f\equiv ax^2+by^2+\cdots$; when the degree of f exceeds 2 even partial evaluations of $N(n=f)$ are at present unknown. It seems fair to say that the simplest case of all, $N(n=x^2+by^2)$, $b>0$, is still far from complete; Dirichlet's well known general theorem† for the number of representations by the totality of a system of representative forms of determinant—b

* Presented to the Society, San Francisco Section, April 5, 1924.

† Cf. Dickson's *History*, vol. 3, p. 19. References to the other citations of this introduction can be found by consulting the index to vol. 3, and running down the references to vol. 2.

does not of itself give a complete solution when the principal genus contains more than one class. For $f \equiv ax^2 + by^2 + cz^2$ there is the complete evaluation of $N(n = f)$ in the case $(a, b, c) = (1, 1, 1)$ by Gauss, an unproved statement of Liouville for $(a, b, c) = (1, 2, 3)$, partial results by Torelli for $(a, b, c) = (1, 1, 2)$, a special case of $(a, b, c) = (1, 2, 2)$ by Stieltjes, and beyond these apparently nothing detailed and specific for this $N(n = f)$. When $f \equiv ax^2 + by^2 + cz^2 + dt^2$, there is Jacobi's $N(n = f)$ for $(a, b, c, d) = (1, 1, 1, 1)$, several theorems of Liouville for $a = 1$ and each of b, c, $d = p^\alpha$ ($\alpha = 0, 1$; $p = 3, 5$) times a low power of 2, some similar results by Humbert when $p = 11$, or when $p = 3$, $\alpha = 2$, and Chapelon's evaluations when $p = 5$, $\alpha = 1, 2$. These appear to mark the limit of definite progress in this direction. Complete evaluations of $N(n = ax^2 + by^2 + \cdots)$ for more than 4 indeterminates $x, y, \ldots$ exist only for 5 and 7 squares. These remarks will indicate how far from satisfactory solutions even the simplest problems in the enumerative arithmetic of homogeneous forms still are.

The final formulas of this paper in § 5 have been checked. The nature of the work is such that this verifies all preceding formulas.

2. *Notation.* In all that follows p is prime, the integers n, a, b, c are prime to p, and a, b, c are coprime; k, M, A, B, C are arbitrary integers; α, β, γ, δ are integers $\geqq 0$. To simplify the printing we shall write

$$N(p^\alpha n = ap^\delta x^2 + bp^\beta y^2 + cp^\gamma z^2) \equiv (\alpha; \delta, \beta, \gamma),$$

in which δ, β, γ (also a, b, c, n) are regarded as given constants. Note that $p^\alpha n$ is any integer.

3. *Lemma.* Although it may be obvious that

(1) $N(kM = kAx^2 + kBy^2 + kCz^2) = N(M = Ax^2 + By^2 + Cz^2)$,

we shall prove it, as upon this depends all that follows. The $\sum$ on the left extending to all integers $x, y, z \gtreqless 0$, that on the right to all integers M,

$$\sum q^{kAx^2 + kBy^2 + kCz^2} \equiv \sum q^{kM} N(kM = kAx^2 + kBy^2 + kCz^2).$$

In this replace q by $\sqrt[k]{q}$:

$$\sum q^{Ax^2+By^2+Cz^2} \equiv \sum q^M N(kM = kAx^2+kBy^2+kCz^2).$$

In the original identity take $k=1$:

$$\sum q^{Ax^2+By^2+Cz^2} \equiv \sum q^M N(M = Ax^2+By^2+Cz^2).$$

By comparing the second identity with the third we get (1).

By the notation explained in § 2, the evaluation of $N(M = Ax^2+By^2+Cz^2)$ is equivalent to that of $(\alpha; \delta, \beta, \gamma)$. By the lemma, if $\alpha < \delta, \beta, \gamma, (\alpha; \delta, \beta, \gamma) = 0$; if $\alpha \geqq \delta, (\alpha; \delta, \delta, \delta) = (\alpha-\delta; 1, 1, 1)$; while if δ, β, γ are unequal, one of them, say δ, is not greater than either of the others, and if $\alpha \geqq \delta$, $(\alpha; \delta, \beta+\delta, \gamma+\delta) = (\alpha-\delta; 0, \beta, \gamma)$.

Hence the evaluation of $N(M = Ax^2+By^2+Cz^2)$ is reduced to that of $(\alpha; 0, \beta, \gamma)$, in which, without loss of generality, we may assume $\gamma \geqq \beta$. Evidently the inequality $\gamma \geqq \beta$ (by the definitions of β, γ in § 2) can be eliminated by replacing γ by $\gamma+\beta$ wherever γ occurs. Eliminations of this kind simplify the final formulas. The further evaluation of $N(M = Ax^2+By^2+Cz^2)$ is now reduced to that of $(\alpha; 0, \beta, \beta+\gamma)$.

4. *Preliminary Reductions.* Let $s \geqq 0$ be an integer such that $\alpha-2s$, $\beta-2s \geqq 0$, and therefore also $\beta+\gamma-2s \geqq 0$. Suppose for a moment that for some $s>0$ we have $\alpha-2s$, $\beta-2s>0$. If $(\alpha; 0, \beta, \beta+\gamma)>0$, then must $x \equiv 0 \bmod p$, and therefore by s applications of the Lemma (§ 3),

$$(2) \quad (\alpha; 0, \beta, \beta+\gamma) = (\alpha-2s; 0, \beta-2s, \beta+\gamma-2s),$$

which obviously remains true when $s=0$ and when $(\alpha; 0, \beta, \beta+\gamma) = 0$. Choose for s the lesser of $[\alpha/2]$, $[\beta/2]$, where $[t]$ is the greatest integer $\leqq t$; when $\alpha = \beta$, take $s = [\beta/2]$. Clearly the reductions (2) can be performed precisely s times, s being as just chosen. Separating out the cases of (2) for even and odd values of β we get

$$(1.1) \quad \alpha \leqq \beta, \ (2\alpha; 0, 2\beta, 2\beta+\gamma) = (0; 0, 2\beta-2\alpha, 2\beta+\gamma-2\alpha);$$

$$(1.2) \quad \alpha \geqq \beta, \ (2\alpha; 0, 2\beta, 2\beta+\gamma) = (2\alpha-2\beta; 0, 0, \gamma);$$

(1.3) $\alpha \leqq \beta$, $(2\alpha+1;\ 0,\ 2\beta,\ 2\beta+\gamma)$
$= (1;\ 0,\ 2\beta-2\alpha,\ 2\beta+\gamma-2\alpha)$;
(1.4) $\alpha \geqq \beta$, $(2\alpha+1;\ 0,\ 2\beta,\ 2\beta+\gamma) = (2\alpha+1-2\beta;\ 0,\ 0,\ \gamma)$;
and the complementary set,
(2.1) $\alpha \leqq \beta$, $(2\alpha;\ 0,\ 2\beta+1,\ 2\beta+\gamma+1)$
$= (0;\ 0,\ 2\beta+1-2\alpha,\ 2\beta+\gamma+1-2\alpha)$;
(2.2) $\alpha \geqq \beta$, $(2\alpha;\ 0,\ 2\beta+1,\ 2\beta+\gamma+1)$
$= (2\alpha-2\beta;\ 0,\ 1,\ \gamma+1)$;
(2.3) $\alpha \leqq \beta$, $(2\alpha+1;\ 0,\ 2\beta+1,\ 2\beta+\gamma+1)$
$= (1;\ 0,\ 2\beta+1-2\alpha,\ 2\beta+\gamma+1-2\alpha)$;
(2.4) $\alpha \geqq \beta$, $(2\alpha+1;\ 0,\ 2\beta+1,\ 2\beta+\gamma+1)$
$= (2\alpha+1-2\beta;\ 0,\ 1,\ \gamma+1)$.

Only those on the right having a pair of zeros in the symbol are irreducible. The further reduction of the rest is effected in a similar way, first powers of the prime p, instead of second, being now successively eliminated. The process is seen by examining the right of (1.3), (2.3). When $\alpha < \beta$ we have $2\beta-2\alpha \geqq 2$, $2\beta+\gamma-2\alpha \geqq 2$, and since pn is the number represented in the right of (1.3), it follows that $x \equiv 0 \bmod p$. Applying the lemma, we get
$(1;\ 0,\ 2\beta-2\alpha,\ 2\beta+\gamma-2\alpha)$
$= (0;\ 1,\ 2\beta-2\alpha-1,\ 2\beta+\gamma-2\alpha-1)$,
and this evidently vanishes (when $\alpha < \beta$). Similarly for (2.3), and we have
(1.31) $\alpha < \beta$, $(2\alpha+1;\ 0,\ 2\beta,\ 2\beta+\gamma) = 0$;
(2.31) $\alpha < \beta$, $(2\alpha+1;\ 0,\ 2\beta+1,\ 2\beta+\gamma+1) = 0$,
which may replace (1.3), (2.3), since the cases $\alpha = \beta$ are included in (1.4), (2.4).

Similarly, provided that $\alpha-2s,\ \gamma-2s+1 \geqq 0$, we get

$$(\alpha;\ 0,\ 1,\ \gamma+1) = (\alpha-2s;\ 0,\ 1,\ \gamma+1-2s),$$

and, provided that $\alpha-1-2s,\ \gamma-2s \geqq 0$,

$$(\alpha;\ 0,\ 1,\ \gamma+1) = (\alpha-1-2s;\ 1,\ 0,\ \gamma-2s).$$

Upon separation of cases according to even, odd γ, these yield the formulas which enable us to complete the re-

duction of (1. 1)—(2. 4). It is unnecessary to preserve the very simple calculations. We find

(3. 1) $\alpha \leqq \gamma+1$, $(2\alpha;\ 0,\ 1,\ 2\gamma+2) = (0;\ 0,\ 1,\ 2\gamma+2-2\alpha)$;
(3. 2) $\alpha \geqq \gamma+1$, $(2\alpha;\ 0,\ 1,\ 2\gamma+2) = (2\alpha-2\gamma-2;\ 0,1,0)$;
(3. 3) $\alpha \leqq \gamma$, $(2\alpha+1;\ 0,\ 1,\ 2\gamma+2) = (0;\ 1,\ 0,\ 2\gamma+1-2\alpha)$;
(3. 4) $\alpha \geqq \gamma+1$, $(2\alpha+1;\ 0,\ 1,\ 2\gamma+2) = (2\alpha-2\gamma-1;\ 0,1,0)$,

and the complementary set

(4. 1) $\alpha \leqq \gamma$, $(2\alpha;\ 0,\ 1,\ 2\gamma+1) = (0;\ 0,\ 1,\ 2\gamma+1-2\alpha)$;
(4. 2) $\alpha \geqq \gamma+1$, $(2\alpha;\ 0,\ 1,\ 2\gamma+1) = (2\alpha-2\gamma-1;\ 1,\ 0,\ 0)$;
(4. 3) $\alpha \leqq \gamma$, $(2\alpha+1;\ 0,\ 1,\ 2\gamma+1) = (0;\ 1,\ 0,\ 2\gamma-2\alpha)$;
(4. 4) $\alpha \geqq \gamma$, $(2\alpha+1;\ 0,\ 1,\ 2\gamma+1) = (2\alpha-2\gamma;\ 1,\ 0,\ 0)$,

all of which are further irreducible. Note that since γ may take the value zero, $(\alpha;\ 0,\ 1,\ 2\gamma)$ is not necessarily reducible, while the type considered, $(\alpha;\ 0,\ 1,\ 2\gamma+2)$, is.

Apply (3. 1)—(4. 4) to (2. 2), (2. 4) after having first eliminated the condition $\alpha \geqq \beta$ by replacing α wherever it occurs by $\beta+\alpha$. The results are:

(5. 1) $\alpha \leqq \gamma$, $(2\beta+2\alpha;\ 0,\ 2\beta+1,\ 2\beta+2\gamma+1) = (0;\ 0,\ 1,\ 2\gamma+1-2\alpha)$;
(5. 2) $\alpha \geqq \gamma+1$, $(2\beta+2\alpha;\ 0,\ 2\beta+1,\ 2\beta+2\gamma+1) = (2\alpha-2\gamma-1;\ 1,\ 0,\ 0)$;
(5. 3) $\alpha \leqq \gamma$, $(2\beta+2\alpha+1;\ 0,\ 2\beta+1,\ 2\beta+2\gamma+1) = (0;\ 1,\ 0,\ 2\gamma-2\alpha)$;
(5. 4) $\alpha \geqq \gamma$, $(2\beta+2\alpha+1;\ 0,\ 2\beta+1,\ 2\beta+2\gamma+1) = (2\alpha-2\gamma;\ 1,\ 0,\ 0)$;

and the complementary set,

(6. 1) $\alpha \leqq \gamma+1$, $(2\beta+2\alpha;\ 0,\ 2\beta+1,\ 2\beta+2\gamma+2) = (0;\ 0,\ 1,\ 2\gamma+2-2\alpha)$;
(6. 2) $\alpha \geqq \gamma+1$, $(2\beta+2\alpha;\ 0,\ 2\beta+1,\ 2\beta+2\gamma+2) = (2\alpha-2\gamma-2;\ 0,\ 1,\ 0)$;
(6. 3) $\alpha \leqq \gamma$, $(2\beta+2\alpha+1;\ 0,\ 2\beta+1,\ 2\beta+2\gamma+2) = (0;\ 1,\ 0,\ 2\gamma+1-2\alpha)$;
(6. 4) $\alpha \geqq \gamma+1$, $(2\beta+2\alpha+1;\ 0,\ 2\beta+1,\ 2\beta+2\gamma+2) = (2\alpha-2\gamma-1;\ 0,\ 1,\ 0)$.

Examining (1. 1)—(2. 4) and (5. 1)—(6. 4) we see it is necessary to consider only

$$(7)\qquad N(p^{\alpha}n = ax^2 + bp^{2\beta}y^2 + cp^{2\beta+\gamma}z^2) \equiv N_1(p^{\alpha}n),$$

$$(8)\qquad N(p^{\alpha}n = ax^2 + bp^{2\beta+1}y^2 + cp^{2\beta+2\gamma+1}z^2) \equiv N_2(p^{\alpha}n),$$

$$(9)\qquad N(p^{\alpha}n = ax^2 + bp^{2\beta+1}y^2 + cp^{2\beta+2\gamma+2}z^2) \equiv N_3(p^{\alpha}n),$$

in order, by the reductions in § 3, to obtain a complete set of reduction formulas for

$$(10)\qquad N(M = Ax^2 + By^2 + Cz^2).$$

That the three sets in § 5 are exhaustive is evident by inspection on referring to the notation in § 2.

5. *Final Formulas.* For the N_i $(i = 1, 2, 3)$ see (7), (8), (9). From (1. 1)—(1. 4) and (1. 31), by eliminating the condition $\alpha \geqq \beta$, we find

(I) *Form* $ax^2 + bp^{2\beta}y^2 + cp^{2\beta+\gamma}z^2$:

$$\alpha \leqq \beta,\ N_1(p^{2\alpha}n) = (0;\ 0,\ 2\beta - 2\alpha,\ 2\beta + \gamma - 2\alpha);$$
$$\alpha < \beta,\ N_1(p^{2\alpha+1}n) = 0;$$
$$N_1(p^{2\beta+\alpha}n) = (\alpha;\ 0,\ 0,\ \gamma).$$

From (2. 1), (2. 31) with γ replaced by 2γ, and from (5. 1)—(5. 4) we find upon eliminating $\alpha \geqq \gamma + 1$, $\alpha \geqq \gamma$.

(II) *Form* $ax^2 + bp^{2\beta+1}y^2 + cp^{2\beta+2\gamma+1}z^2$:

$$\alpha \leqq \beta,\ N_2(p^{2\alpha}n) = (0;\ 0,\ 2\beta + 1 - 2\alpha,\ 2\beta + 2\gamma = 1 - 2\alpha);$$
$$\alpha < \beta,\ N_2(p^{2\alpha+1}n) = 0;$$
$$\alpha \leqq \gamma,\ N_2(p^{2\beta+2\alpha}n) = (0;\ 0,\ 1,\ 2\gamma + 1 - 2\alpha);$$
$$N_2(p^{2\beta+2\gamma+2\alpha+2}n) = (2\alpha + 1;\ 1,\ 0,\ 0);$$
$$\alpha \leqq \gamma,\ N_2(p^{2\beta+2\alpha+1}n) = (0;\ 1,\ 0,\ 2\gamma - 2\alpha);$$
$$N_2(p^{2\beta+2\gamma+2\alpha+1}n) = (2\alpha;\ 1,\ 0,\ 0).$$

From (2. 1), (2. 31) with γ replaced by $2\gamma + 1$, and from (6. 1)—(6. 4) upon elimination of $\alpha \geqq \gamma + 1$ we find

(III) *Form* $ax^2 + bp^{2\beta+1}y^2 + cp^{2\beta+2\gamma+2}z^2$:

$$\alpha \leqq \beta,\ N_3(p^{2\alpha}n) = (0;\ 0,\ 2\beta + 1 - 2\alpha,\ 2\beta + 2\gamma + 2 - 2\alpha);$$
$$\alpha < \beta,\ N_3(p^{2\alpha+1}n) = 0;$$

$\alpha \leqq \gamma + 1$, $N_3(p^{2\beta+2\alpha}n) = (0; 0, 1, 2\gamma + 2 - 2\alpha)$;

$N_3(p^{2\beta+2\gamma+2\alpha+2}n) = (2\alpha; 0, 1, 0)$;

$\alpha \leqq \gamma$, $N_3(p^{2\beta+2\alpha+1}n) = (0; 1, 0, 2\gamma + 1 - 2\alpha)$;

$N_3(p^{2\beta+2\gamma+2\alpha+3}n) = 2\alpha + 1; 0, 1, 0)$.

In all of the above no further reduction is possible.

6. *Successive Reductions.* Let D be the greatest common divisor of B, C in (10), and assume without loss of generality (§ 3) that M, A, B, C are relatively prime in their totality. Let $M = M'p^\alpha$, where p is any prime divisor of D, and M' is prime to p. Apply (I)—(III) of § 5. Repeat the process on the results for each remaining prime divisor of D, obtaining finally a system of formulas analogous to (I)—(III) in which (10), for its several possible cases according to the prime factors of D, is replaced by a corresponding $N(M'p^{\alpha'} = A'x^2 + B'y^2 + C'z^2)$ in which no further reduction with respect to B', C' is possible. This system of formulas may conveniently be written as a set of equalities between r-rowed matrices, where r is the number of distinct prime factors of D. To each pair of A, B, C in (10) will correspond such a system of equalities, and all three together give the complete reduction of (10). It would be of interest to discuss this set.

7. (IV) *Form* $Ax^2 + By^2 + Cz^2 + \cdots + Et^2$. As in § 3 the reduction for this form is referred to that of

$$N(p^\alpha n = x^2 + p^\beta y^2 + p^{\beta+\gamma} z^2 + \cdots + p^{\beta+\gamma+\cdots+\varepsilon} t^2),$$

where $\beta, \gamma, \ldots, \varepsilon$ are integers $\geqq 0$, and a precisely similar argument shows immediately that this N is reduced when $N(p^\alpha n = x^2 + p^\beta y^2 + p^{\beta+\gamma} z^2)$ is reduced. The complete set of reduction formulas can be written down from § 5.

8. *General Form.* When the degree of f is $3 + \alpha$, the process of reducing $N(n = f)$ is evident from the foregoing; the discussion now depends upon $[k/(3 + \alpha)]$ instead of $[k/2]$.

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THE SCIENTIFIC WORK OF JOSEPH LIPKA

BY W. C. GRAUSTEIN

Joseph Lipka, Associate Professor of Mathematics at the Massachusetts Institute of Technology, died January 15, 1924, in his forty-first year and the sixteenth year of his service at the Institute. He was a mathematician of proven worth, intensely interested in his science, and unremitting in his efforts to forward its progress. The last few years of his life were especially remarkable for their productivity. During this time he contributed what would normally be considered, both in volume and value, as the creditable work of a decade, and gave abundant promise of continued fruitfulness. That he has been taken in his prime is indeed a source of great regret and a distinct loss to mathematics.

During Lipka's student days at Columbia, Professor Kasner was busily applying differential geometry to dynamics and developing, in euclidean spaces of two and three dimensions, the geometric properties of dynamical trajectories and related systems of curves. It was this field of investigation in which Lipka was later initiated and did his work for the doctorate. His thesis consisted in a generalization, to a curved space of n dimensions, of results obtained by Kasner concerning a natural family of curves, that is, a family of trajectories in a conservative field of force corresponding to a given constant of energy, or more generally, any family of extremals resulting from a problem in the calculus of variations of the form,

$$\int e^{\varphi} ds = \text{minimum}, \tag{1}$$

where ds is the linear element of the space in question, and φ is a function of the coordinates of the space.

Lipka's later work, though largely confined to the one field, falls naturally into two periods. The earlier of these sees the generalization of many of Kasner's theories, first to ordinary surfaces and then to curved spaces of n dimensions. Noteworthy is the exhaustive study of dynamical trajectories on a surface for any positional field of force, and also the investigation of the geometrical properties of trajectory and related systems in Riemannian n-space. But perhaps the most important achievement of this period was the paper which established the validity, in all cases, of the Thomson-Tait criterion for a natural family. According to the theorem of Thomson and Tait, the ∞^{n-1} curves of a natural family meeting an arbitrary hypersurface orthogonally form a normal hypercongruence, i. e., admit of ∞^1 normal hypersurfaces. That this property is characteristic of natural familes (for $n > 2$) had been proved by Kasner for a euclidean space of three dimensions. In the paper now

under discussion, this was proved for a euclidean space, first of four and then of n dimensions, and finally for a general curved space.

During his career Lipka made two trips abroad, and both of them he turned to good account. The first, during the summer of 1913, brought him into contact with Professor Whittaker and the mathematical laboratory at Edinburgh and thus furnished the inspiration which led him to found at the Institute a laboratory course and later to write the text on *Graphical and Mechanical Computation*.

The summer of 1921 and the following academic year Lipka spent on leave in Europe, principally in Rome. This sojourn abroad marked the beginning of a new period in his scientific work. The stimulus and inspiration which he derived from his new associations, especially from those with Professor Levi-Civita, are clearly evident in his writings. His work is now permeated with the methods of the absolute calculus and governed largely by new ideas. As a first illustration may be mentioned the use of Hamilton's canonical equations and infinitesimal contact transformations to prove anew, in a brief and elegant fashion, the Thomson-Tait characteristic property of natural families, and to extend this property so that it applies to an irreversible system; in the extension, which of course applies merely to a conservative field of force, it is no longer the curve which is orthogonal to the hypersurfaces in question, but the resultant of the velocity vector along the curve and the vector whose components are the coefficients of the (new) linear term in the Lagrangian function.

Another group of papers revolve about Levi-Civita's notion of parallelism, or parallel motion of a direction along a curve, in a Riemannian space. By use of this concept Lipka gives a simple intrinsic definition of geodesic curvature, which is the analogue of that of ordinary curvature in euclidean space. Thus, if P' is a point on a curve C neighboring to a given point P of C, and if $d\omega$ is the angle between the tangent at P' and the parallel at P' to the tangent at P, the limit of the ratio of $d\omega$ to the arc PP' is the geodesic curvature of C at P. In a later paper a similar process is employed to obtain the curvature of C at P relative to any curve C' tangent at P to C. This new type of relative curvature finds application in dynamics when the curve C' is taken as the trajectory of a natural family.

From equation (1) it is clear that natural families are a generalization of geodesics and that every natural family is the conformal representation of the geodesics in a second space. It was doubtless these facts which led Lipka to generalize Levi-Civita's concept of parallelism by replacing the geodesics at the basis of it by the trajectories of a natural family, and which suggested for the new type of parallelism the name "conformal parallelism". This idea, by its very nature, is of value in the study of dynamical problems. For example, a trajectory

is now characterized by the fact that its direction at any point is conformally parallel to its initial direction. Again, the curves orthogonal to the ∞^1 trajectories on an ordinary surface which issue at right angles from a given curve are conformally parallel.

Perhaps the aspect of conformal parallelism which will make the widest appeal is its close connection with the second differential parameter. Levi-Civita has shown that the Gaussian curvature of a surface at a point P is the ratio of the angle between the initial and final positions of a direction which moves by parallelism completely around an infinitesimal cycle through P, to the area of the cycle. Lipka shows that, in the case of conformal parallelism, this ratio is equal to the difference between the curvature at P and the value at P of $\Delta_2\varphi$, where φ is the "characteristic" function given in (1). If, then, φ is any point function on the surface, the value at P of $\Delta_2\varphi$ can be interpreted as the ratio of the angle between the two final positions of a direction which moves first by parallelism and then by conformal parallelism (with the characteristic function φ) completely around the cycle, to the area of the cycle.

The substitution of trajectories for geodesics and trajectory surfaces for geodesic surfaces forms also the basis for the development, in another paper, of a generalization of the Riemannian theory. Thereby Riemannian curvature is replaced by trajectory curvature, that is, the Gaussian curvature of a trajectory surface at a point. Moreover, the theory developed by Ricci concerning principal directions and principal curvatures lends itself to complete generalization. The new theory contains the old as a special case, when φ is constant. The relationships between the fundamental quantities in the two theories are thus readily obtained and are quite remarkable in their simplicity and elegance.

Lipka's last published paper has to do with a system of invariants introduced by Ricci in his study of orthogonal congruences, and called by him coefficients of rotation, in light of the fact that he was able to show that they can be interpreted in terms of rotations in the euclidean space in which the given curved space is imbedded. Following a suggestion of Levi-Civita, Lipka proves that these invariants admit a simple intrinsic interpretation within the given space by means of the notion of parallelism. This interpretation enables him to give elegant characterizations of certain types of congruences, notably congruences of geodesics, in terms of the vanishing of certain of the invariants, and to obtain other results of equal importance. Finally, by introducing conformal parallelism instead of that of Levi-Civita, he develops a new set of invariants, which yield corresponding results bearing on trajectories.

It is a pleasure to read Lipka's writings. The habit of clear thinking and clear expression was ingrained in him. To this, as well as to his personal qualities, was due his success as a teacher.

Lipka was natural, unpretentious, always cheerful and sympathetic,

ever willing to lend a hand, and enthusiastically devoted to his chosen work. And beneath these traits there was a definite, but more elusive, quality which vitalized them, a certain inner buoyancy, an ever forward pressing optimism, an intangible force which drew others to him.

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1918

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1920

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On conformal parallelism. JOURNAL OF MATHEMATICS AND PHYSICS, Massachusetts Institute of Technology, vol. 2, No. 3, pp. 175–194.

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On the relative curvature of two curves in V_n. This BULLETIN, vol. 29, No. 8, pp. 345–348.

1924

On Ricci's coefficients of rotation. JOURNAL OF MATHEMATICS AND PHYSICS, Massachusetts Institute of Technology, vol. 3, No. 1, pp. 7–23.

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VEBLEN ON ANALYSIS SITUS

The Cambridge Colloquium, 1916, Part II. *Analysis Situs.* By Oswald Veblen. New York, The American Mathematical Society, 1922. vii + 150 pp.

The present reviewer finds himself facing the infrequent experience of tackling the same work for the second time. That he derives no little satisfaction from it, is due to the fact that since the first review was written* he has made ample use of the book as an instrument of research, and hopes that the additional experience thus acquired will prove of value not only to himself but also to his readers.

Let us say at once that the *Lectures* have stood the test of usage very well indeed. Hitherto a beginner attracted by the fascinating and difficult field of analysis situs, was obliged to wade through many widely scattered papers, beginning with Poincaré's classic of the 1895 in JOURNAL DE L'ÉCOLE POLYTECHNIQUE and its multiple *Compléments.* Difficult reasonings beset him at every step, an unfriendly notation did not help matters, to all of which must be added, most baffling of all, the breakdown of geometric intuition precisely when most needed. No royal road can be created through this dense forest, but a good and thoroughgoing treatment of fundamentals, notation, terminology, may smooth the path somewhat. And this and much more we find supplied by Veblen's *Lectures.* That few, if any, were better qualified than the author, by temperament and scientific past, to produce such a work, is well known among the devotees of analysis situs, whose number it will assuredly increase. As this field presents difficult problems in profusion, with but few general methods, such an increment is very much to be hoped for.

Two streams are found in analysis situs: the one related to point-set theory, the other primarily of a combinatorial nature, wherein are treated manifolds which for example in the case of two dimensions, are depictable upon polyhedra with a finite number of plane polygonal faces. To this latter branch these *Colloquium Lectures* are almost wholly devoted. Withal the author is perhaps at his mathematical best at the points of contact of the two streams, say with the type of question presented by the occurrence of Jordan curves on a representative polyhedron.

Veblen has made extensive and very systematic use of the Poincaré

* See *Bulletin des Sciences Mathématiques* for 1922.

† A k-cell is any point-set continuously depictable upon the interior of a hypersphere in k-space.

matrices of a manifold, whereby the incidence relations beween the k-cells and $(k-1)$-cells† are defined by the elements of a certain matrix. These elements are $0, \pm 1$, when the orientations of the cells are taken into consideration. In a notable paper (ANNALS OF MATHEMATICS, 1913) Veblen and Alexander showed that for many purposes the elements could be taken merely modulo 2. By comparing all these matrices and reducing them in the usual manner, one obtains the all important *topological invariants* and their relations. Our geometer friends will ruefully complain that the structure is much too analytical for their taste, but there is no help to it for the present, as nothing else comparably rigorous is now in print. To the above aspect of the theory are devoted four of the five Lectures. Of particular excellence are the elegant development of normal forms for two-dimensional manifolds, the treatment of questions of orientation, also of *covering complexes*,—one of the most useful features of the book. Everything pertaining to general manifolds immersed in a given one with assigned cellular structure is thereby thoroughly clarified. Of decided interest in this connection is the author's reduction of arbitrary k-circuits (k-cycles) on a manifold to those composed exclusively of k-cells of a particular cellular subdivision. Thus a Jordan curve on a polyhedron would be reduced to a polygon of edges. From this follows indeed that certain integers related to the representative matrices (Betti numbers, torsion-coefficients, etc.) are actually topological invariants.

The fifth and last Lecture differs in type from the others, and is more in the nature of what we have been accustomed to expect of such Lectures. It contains an excellent summary of several important questions: homotopy and isotopy, theory of the indicatrix, a fairly ample treatment of the group of a manifold, finally a bird's eye view of what is known and not known (mostly the latter) on three dimensional manifolds.

There is no fundamental criticism to level at the *Lectures*. An error in the argument pp. 142-143 has been pointed out by J. Nielsen of Copenhagen. And if one has some good important theorems, why refrain from displaying them appropriately by means of heavy type, titles, and the like? Of this sin the author is repeatedly guilty: it does not facilitate the beginner's task. We should also have been most thankful for a more complete bibliography.

These *Colloquium Lectures*, one of the best productions of a Series of which American mathematics may well be proud, deserve to be and will be well received by fellow workers in this field and by the general mathematical public.

S. LEFSCHETZ

GOURSAT ON PFAFF'S PROBLEM

Leçons sur le Problème de Pfaff. By Edouard Goursat, Paris, J. Hermann, 1922. 386 pp.

On the first page of Volume 3 of the *Institutiones Calculi Integralis*, Euler declares with reference to the equation $Pdx + Qdy + Rdz = 0$, that there must be some multiplier M by means of which the expression on its left-hand side will become an exact differential; "for unless such a multiplier existed, the proposed differential equation would become absurd and would mean nothing at all". If this opinion had remained unchallenged, the volume under review would not have had much reason for existence. For the problem of Pfaff with which it is occupied relates to differential expressions of which the linear form $\sum_{i=1}^{n} X_i(x_1, \cdots, x_n)\,dx_i$ is the simplest type, independently of whether or not there exists a multiplier which renders them exact.

In a memoir of 1814, J. F. Pfaff showed the essential equivalence of the partial differential equation of the first order

$$\frac{dz}{dx_1} = \psi\left(z, x_1, \cdots, x_n, \frac{dz}{dx_2}, \cdots, \frac{dz}{dx_n}\right)$$

and of the total differential equation

$$dz - \psi(z, x_1, \cdots, x_n, p_2, \cdots, p_n)dx_1 - p_2dx_2 - \cdots - p_ndx_n = 0.$$

A solution of the latter equation consists of a set of n equations in $z, x_1, \cdots, x_n, p_2, \cdots, p_n$, which are solvable for $z, p_2, \cdots, p_n$ in terms of $x_1, \cdots, x_n$ and thus yield functions which reduce this equation to an identity in $x_1, \cdots, x_n$ and their differentials. It is a special case of the general equation $X_1dx_1 + \cdots + X_ndx_n = 0$, in which X_i are functions of $x_1, \cdots, x_n$. The reduction of this equation and of the differential form on its left-hand side has been the object of many writers, among whom are Gauss, Jacobi, Grassmann, Clebsch, Darboux, Frobenius. This question occupies nearly all of Part I of Forsyth's *Theory of Differential Equations*, to which M. Goursat refers for a historical account of the development of the subject. It also furnishes the material for E. von Weber's *Vorlesungen über das Pfaffsche Problem*, to which only very casual reference is made in M. Goursat's volume. Indeed the latter is very different in character, for it is not primarily concerned with an account of the methods for the solution of the problem developed by various writers, but rather with its extensions and generalizations. This indeed is its real reason for existence as a separate volume.

The first two chapters form a natural sequel to the author's *Leçons*

sur l'Intégration des Équations aux Dérivées Partielles du Premier Ordre to which they are similar in style. In the first, we find a thorough treatment of the fundamental theorem of Pfaff on the reduction of the linear differential form $\sum_1^n X_i dx_i$ to one or the other of the canonical forms $z_1dy_1 + \cdots + z_pdy_p + dy_{p+1}$ or $z_1dy_1 + \cdots + z_pdy_p$ according as the class of the form is $2p+1$ or $2p$, where $z_1, \cdots, z_p, y_1, \cdots, y_{p+1}$ are linearly independent functions of $x_1, \cdots, x_n$. The concept "class of a form" introduced by Frobenius and the reductions to the canonical forms are developed here by the very elegant methods of Frobenius and Darboux, which are based on a consideration of a bilinear covariant of ω, viz.

$$\omega' = \sum a_{ij} dx_i \delta x_j,$$

where

$$a_{ij} = \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i},$$

and of four associated covariant linear differential systems. The matrix of the a_{ij} being skew symmetric, it becomes apparent that the even- or oddness of the class c will be of importance. This distinction does indeed run throughout the whole theory of the forms of Pfaff. A good knowledge of completely integrable linear differential systems is necessary to understand this part of the book.

Chapter 2 is devoted to the problem from which the book derives its title, viz., that of solving the equation $\omega = 0$. A solution consists of p equations $f_i(x_1, \cdots, x_n) = 0$, $i = 1, \cdots, p$, $p < n$, such that from them together with those obtained from them by differentiation, the given equation follows. It is shown that the maximum number of dimensions of a manifold which in this sense satisfies the equation is $n - (\gamma + 1)/2$, where γ is the class of the equation, a number which is equal to c or to $c-1$, according as c is odd or even; every integral manifold of lower dimensionality lies in this maximum-dimensional manifold; moreover, the maximum number of dimensions is never less than the integral part of $n/2$. The families of characteristic manifolds are determined by the γ solutions of one of the associated complete linear differential systems referred to above, and a general theorem is obtained establishing their relation to the integral manifolds.

The second division of the book consisting of Chapters 3, 4, and 5 deals with the generalization of the problem of Pfaff to differential forms of higher degree

$$\Omega \equiv \sum A_{\alpha_1 \ldots \alpha_p} dx_{\alpha_1} \ldots dx_{\alpha_p}$$

in which $dx_{\alpha_1} \ldots dx_{\alpha_p}$ is a symbolic product whose characteristic properties are to be determined. To justify this problem as an extension

of the problem of Pfaff, the latter is restated in the following form: To determine in n-space all the r-manifolds ($1 \leqq r < n$) such that the line integral $\int \omega$ shall vanish if taken along any curve on such a manifold. The generalized problem is then to determine in n-space all r-manifolds $p \leqq r < n$ such that the $\int \Omega$ vanishes if taken over any p-manifold situated on them. The theory of the symbolic product $dx_{a_1} \ldots dx_{a_p}$ and of the symbolic forms Ω, as developed and applied in Chapters 3 and 4 is largely the work of MM. Cartan and Goursat. As remarked in the preface, the study of the symbolic products could readily be connected with the tensor calculus. It seems to the reviewer indeed a wise decision to have given the development independently of the general theory.

The bilinear covarient ω' introduced for the simple case by Darboux and Frobenius finds an extension in the derivative Ω' of the form Ω. As in the linear problem the vanishing of ω' was the necessary and sufficient condition for exactness, so $\Omega' = 0$ is the necessary and sufficient condition for the exactness of Ω. The concept "class of a form" is extended as are also the associated differential systems.

Of especial interest in connection with the results of Chapters 1 and 2 are the applications of the symbolic forms to the equation of Pfaff. The p^{th} derivative of a linear form ω being a symbolic form of degree $p+1$, defined by the recursion formulas

$$\omega^{(2m-1)} = \frac{1}{m!}(\omega')^m,$$

$$\omega^{(2m)} = \omega(\omega')^m,$$

it is found that if the c^{th} derivative is the first among the successive derivatives of a linear form ω which vanishes, then the class of ω is equal to c. The derivative forms furnish moreover in a very direct way a method for the reduction of a form of Pfaff to the canonical form discussed in Chapters 1 and 2. Chapter 4 closes with the application of these results to a system of partial differential equations and to the problem of contact transformations.

The fundamental properties of the integral invariants of a system of differential equations

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n} = dt$$

are shown in Chapter 5 to follow quite naturally from the properties of the symbolic forms considered in the preceding chapter. The treatment followed complements in an interesting manner that of Cartan in his recent book on Integral Invariants (see this BULLETIN, vol. 29, p. 140). There are a variety of beautiful results: the relation between the invariant character of $\int \omega$ and of $\int \omega'$, ω being a form of any degree; the theorem

that every relative integral invariant is the sum of an absolute integral invariant and the integral of a symbolic total differential, etc.

The last three chapters are devoted to recent results in the study of systems of Pfaff, a great part of which is due to Cartan. The first question considered in Chapter 6 is as to the maximum number of dimensions ρ of an integral manifold of a system of r forms of Pfaff.

There is a rich amount of material, too varied to lend itself to brief discussion, on systems of linear differential expressions, contact transformations, derived systems, the problem of Monge, second order partial differential equations, etc. The final chapter is devoted to the classification of the integral elements of a system of Pfaff and to the existence theorem.

The book affords an excellent introduction to the study of a problem which occupies a very central position in the theory of differential equations. It gives a detailed survey of the classical theory, of its connections with other domains, of its most modern delelopments, and of the directions in which further advances may be made.

A. Dresden

WATSON ON BESSEL FUNCTIONS

A Treatise on the Theory of Bessel Functions. By G. N. Watson. Cambridge, University Press, 1922. viii + 804 pages.

The purpose of this book is twofold: to develop certain applications of the fundamental processes of the theory of functions of complex variables for which Bessel functions are admirably adapted; and secondly, to compile a collection of results which shall be of value to the increasing number of mathematicians and physicists who encounter Bessel functions in the course of their researches. The author believes that the existence of such a collection is demanded by the greater abstruseness of properties of Bessel functions (especially of functions of large order) which have been required in recent years in various problems of mathematical physics.

In his exposition the author has endeavoured to accomplish two specific results: to give an account of the theory of Bessel functions which a pure mathematician would regard as fairly complete; and to include all formulas, whether general or special, which, although without theoretical interest, are likely to be required in practical applications. An attempt is made to give the latter results, as far as possible, in a form appropriate for use in the applications. These exalted aims the author seems to have achieved with a remarkable success. The

great amount of material thus to be included and the necessity for keeping the size of the book within bounds have made necessary the employment of a concise and compact exposition: but this has been attained without undue sacrifice of clarity. Nearly all parts of the book are as easily read as one has a right to expect in the case of material of this sort. The general theory on which the special results in this volume are based is to be found in the *Course of Modern Analysis* by Whittaker and Watson. It is greatly to the reader's convenience to have a single volume for reference for the basic theory on which the exposition is founded.

Concerning the choice of canonical functions the author speaks as follows in his preface: "It is desirable to draw attention here to the function which I have regarded as the canonical function of the second kind, namely the function which was defined by Weber and used subsequently by Schläfli, by Graf and Gubler, and by Nielsen. For historical and sentimental reasons it would have been pleasing to have felt justified in using Hankel's function of the second kind; but three considerations prevented this. The first is the necessity for standardizing the function of the second kind; and, in my opinion, the authority of the group of mathematicians who use Weber's function has greater weight than the authority of the mathematicians who use any other one function of the second kind. The second is the parallelism which the use of Weber's function exhibits between the two kinds of Bessel functions and the two kinds (cosine and sine) of trigonometric functions. The third is the existence of the device by which interpolation is made possible in Tables I and III at the end of Chapter XX, which seems to me to make the use of Weber's function inevitable in numerical work."

In connection with each section references are given to the memoirs in which the results were first announced but the methods of proof employed are often different from those of the original investigators. A very complete bibliography of thirty-six pages is given at the end of the book.

Space does not allow us to give an analysis of the contents of so large a volume. But we may give a bare outline of the contents by means of chapter headings. These are as follows: Bessel functions before 1826 (pages 1–13), the Bessel coefficients (14–37), Bessel functions (38–84), differential equations (85–131), miscellaneous properties of Bessel functions (132–159), integral representations of Bessel functions (160–193), asymptotic expansions of Bessel functions (194–224), Bessel functions of large order (225–270), polynomials associated with Bessel functions (271–307), functions associated with Bessel functions (308–357), addition theorems (358–372), definite integrals (373–382), infinite integrals (383–449), multiple integrals (450–476), the zeros of

Bessel functions (477–521), Neumann series and Lommel's functions of two variables (522–550), Kapteyn series (551–575), series of Fourier-Bessel and Dini (576–617), Schlömilch series (618–653), the tabulation of Bessel functions (654–664), tables of Bessel functions (665–752), bibliography (753–788), index of symbols (789–790), list of authors quoted (791–795), general index (796–804).

The author's work is well done throughout. The reviewer did not detect any omissions or serious blemishes of any kind. In some respects the exposition is probably more disjointed than is necessary. Some parts would profit by being brought into closer connection with related general theories relative to differential equations; but the author is right in avoiding the use of any large part of the general theory of differential equations. It appears to the reviewer that the exposition of the theory of the zeros of Bessel functions would profit by a closer connection with the general oscillation theory of the solutions of linear homogeneous differential equations of the second order, especially since that theory has taken such simple form in the hands of Bôcher. In a similar manner the general notions involved in the Birkhoff theory of expansions in terms of orthogonal and biorthogonal functions satisfying linear differential equations would serve to give greater unity to the four long chapters on expansions in terms of Bessel functions. But, even so, there is something to be said in favor of the direct treatment employed by the author.

Many of the results recorded appear to be of the nature of special instances of general propositions yet to be discovered. In the case of these a disjointed exposition is inevitable until the general theory has been brought into existence. Perhaps the book will serve a third important purpose in addition to the two which the author had in mind in preparing it. It is suited to suggest further researches in two ways: it will raise interesting questions concerning Bessel functions and the differential equation which they satisfy; it will also suggest general theories concerning linear differential equations, theories special instances of which are afforded by the Bessel equation. In this respect the expansion theory seems to hold out the greatest promise. We may well close the review by emphasizing the importance in this respect of the four excellent chapters devoted to expansions of functions in series of Bessel functions.

R. D. CARMICHAEL

BIRKHOFF ON RELATIVITY

Relativity and Modern Physics. By G. D. Birkhoff. Cambridge, Harvard University Press, 1923. xi + 283 pp.

Birkhoff's general plan for his book on relativity is excellent. He begins with a rapid survey of those branches of classical physics which are to be affected most seriously by the new theory. This survey shows that the old theories have been fundamentally limited by the underlying concept of a rigid body in empty space, and leads to the inquiry as to what sort of a theory can be built up by using for our physical model "a number of very small material particles in motion at comparatively great distances from each other in otherwise empty space".

For simplicity he limits attention at first to a one-dimensional case. The position of a particle B with respect to a particle A is determined by the time t_1 of emission and the time t_2 of return of a flash of light from A which is reflected by B.* Thus the events of a one-dimensional universe are seen to constitute a two-dimensional manifold. It is shown how the coordinates t_1 and t_2 may be replaced by the more conventional x, t, but the early part of the discussion is carried out in terms of the physically primordial t_1 and t_2.

By introducing certain assumptions (isometric, etc.) as to the character of space-time, attention is limited to two special cases which Birkhoff calls "aeolotropic" and "isotropic" respectively. The first leads to the classical and the second to the special relativity metric. The consequences of the two assumptions are developed far enough by the end of chapter four to give a clear general idea of the differences between the two cases. From this point on attention is restricted to the isotropic case.

Three chapters are now devoted to the development of the special relativity in a two-dimensional space-time, the main topics being the dynamics of a particle and of a system of particles and one-dimensional hydrodynamics. From this discussion there emerges that complex of ideas which Einstein has bound together in the energy-momentum tensor. Then comes a chapter on tensor analysis and Riemann geometry, followed by a chapter on gravitational theory in two dimensions.

On page 150 (there are 270 pages of text) we turn for the first time to physical problems of more than one space dimension. In two chapters

* This method of arriving physically at a coordinate system had previously been suggested by J. L. Synge, NATURE, vol. 108, p. 275 (October 27, 1921) and worked out to a certain extent for the three-dimensional case.

the previous discussion of special relativity is generalized to four dimensions, so that on arriving at page 186 we have the kinematics, dynamics, and hydrodynamics of the special theory behind us. Reversing the historical order, the author now closes his discussion of the special relativity with an account of the electromagnetic equations.

The remaining sixty odd pages of the book are devoted to the general relativity. They contain first the determination of the possible types of linear homogeneous tensor equations of the second order. Then follows an account of Einstein's partial differential equations for a gravitational field with or without matter. Then the Schwarzschild form for the solar field is deduced and followed by an account of the three famous astronomical applications.

As intimated above, the arrangement of the book seems to the reviewer to be a very happy one. It responds very well to the fact that although the relativity theory arose from a study of the electromagnetic equations it was obtained by revising our views of kinematics and matter. Also, the prolonged attention to the two-dimensional case accustoms the reader to the machinery of tensor analysis and to a long series of abstract conceptions while he is dealing with figures that are easily visualized. The reader is thus prepared to meet the difficulties due to the complicated set of relations in the four-dimensional case, with the most important general concepts already in mind. It is true, on the other hand, as pointed out by Eisenhart in a careful review of this book in *Science* for December 28, 1923, that the methods which suggest themselves first in the two-dimensional case cannot always be used in the four-dimensional one. But such cases are exceptional in this book.

The book is intended as an introduction to the subject which may be used as a text in an undergraduate course. From this point of view, the first four chapters seem very difficult reading. There are other books, as, for instance, that of Kopff, with which it would be easier to start. It seems likely that the book will find its place not as a text-book in the ordinary sense but as one of two or three texts to which the student may refer for different points of view.

The reviewer has heard the opinions of various people who have read Birkhoff's book, or parts of it, without being previously well acquainted with the Einstein theory. In general, the accounts were more favorable from those who were not professional mathematicians than from those who were. The distinction seems to be due to the fact that readers of the latter class were disturbed by what they regarded as lapses from mathematical elegance or clarity of statement. In many cases they were handicapped by their own ignorance of physics. No one in either class regarded the book as easy reading, but all were impressed with the author's originality in point of view and with his grasp of the subject.

The reviewer's own opinion is that the real merit of the book is that it will convey new ideas and stimulating points of view to a reader who has already devoted a good deal of thought to the subject. The derivation of the Schwarzschild form without the hypothesis that the field is static and irreversible is obviously an important contribution. So is the chapter on linear tensor equations. But there are many less prominent matters which show great originality of method or content. The reviewer has dipped into the book at several places and always found something worth thinking about. See, for example, the treatment of statistical mass on page 72, or the form for the equations for gravitation on page 229, or the account of the tensor Φ^{i}_{jkl} (which is in fact the second extension of g_{ij}) on page 123.

The postulational treatment to which the author calls special attention in the preface would better, in the opinion of the reviewer, be characterized as prolegomena to a set of postulates. The term "postulational treatment" has come in recent years to have a very precise significance. It implies an explicit list of undefined terms and a sharp statement of the logical form of the unproved propositions. It usually implies also an investigation of independence proofs. Such a study would doubtless have been impracticable in this book, but it would be interesting to see it carried out on some other occasion along the lines indicated by Birkhoff.

In various places Birkhoff brings out either explicitly or implicitly the great role played by considerations of symmetry and simplicity in the derivation of the equations of physics. One is tempted to feel that many of the other reasons advanced for adopting the expressions actually used in physical theories merely cover up the fact that these expressions are the simplest and most symmetrical ones which do not flagrantly contradict known facts. This reflection is also induced frequently on reading Einstein's book, *The Meaning of Relativity*, and it is doubtless capable and worthy of much more explicit development.

OSWALD VEBLEN

BOREL'S THEORY OF FUNCTIONS.

Méthodes et Problèmes de la Théorie des Fonctions. By Émile Borel. (Collection de monographies sur la théorie des fonctions.) Paris, Gautier-Villars, 1922. ix + 148 pp.

The volume under review, which Borel states in the introduction is to be the last of this justly celebrated collection of monographs to appear under his own name, is designed to supplement the earlier volumes of the series by bringing together in their original form a number of his notes and memoirs on the theory of functions. The twenty-seven articles included in this volume appeared in a number of journals during the years 1895—1912. They cover a wide range of subject matter and indicate clearly the breadth, originality, and versatility which have been the source of the very great influence which Borel has exerted on the progress of analysis in the last thirty years.

The memoirs presented are grouped in four chapters: *Les domaines et la théorie dès ensembles*; *Les opérations et les développements en séries*; *La théorie de la croissance et le rôle des constants arbitraires*; *Les fonctions de variable complexe, en général, et les fonctions particulières.*

The introduction is devoted to an interesting analysis of the many striking analogies between biology and the theory of functions, both in content and in historical order of development. As man learned to use animals and plants before he had acquired a thoroughgoing knowledge of anatomy or physiology so mathematicians discovered and utilized the elementary functions, algebraic, circular, exponential. In biology the study of the structure of organisms led to the analysis of cellular life and organic chemistry; in the theory of functions the analysis of the properties of functions brought into existence the modern theories of aggregates, numbers, sets of points.

In the first chapter Borel remarks that the domains and ensembles are to functions as the tissues are to living organisms and calls attention to the importance which this part of the theory of functions has come to assume. Of the seven papers included in this chapter three are brief notes, one on the representation of discontinuous functions as limits of continuous functions and two on the theory of measure. The last of these outlines a procedure which may be substituted for the method of Lebesgue in the theory of integration. Three articles are devoted to the analysis and classification of sets of measure null. It is significant that the discoverer of the role in analysis of the sets of measure null should be the pioneer in the study of their structure and the closely related theory of monogenic non-analytic functions.

This chapter also contains a memoir entitled, *"Sur les définitions analytiques et sur l'illusion du transfini"*, which presents ably and forcibly the author's views regarding the limitations which should be placed upon arguments involving the transfinite. As is well known, no finite collection of words and symbols is adequate to the representation of all the transfinite ordinals of the second class. Borel calls attention to the relation between these ordinals α and the corresponding types of increasing function $\varphi_\alpha(n)$. In terms of these functions $\varphi_\alpha(n)$ it is possible to define a function $f(x)$ which is not representable analytically, but the complete definition of the function $f(x)$ rests upon the assumed definition of the $\varphi_\alpha(n)$ and the latter definition is impossible in a finite number of words.

The second chapter, on operations and developments in series, contains a short memoir on the conditions governing a change in the order of the terms of a conditionally convergent series, a memoir on the representation of functions of two real variables, and one on the integration of unbounded functions and constructive definitions. In his discussion of constructive definitions Borel expounds in detail his view that the only entities which may properly enter a mathematical discussion are those which are "well-defined". The continuum of Cantor and Dedekind is not "well-defined" in the sense of Borel since the numbers which form the continuum are not all definitely given. Hence, to be entirely consistent, it would be necessary for Borel either to abandon the use of the concept of the continuum and the related theory of functions or else so to transform these theories that they rest upon a "well-defined" basis. In spite of the admonishments of Borel and other adherents of his views, the number of those who call themselves mathematicians and who devote themselves to the study of concepts which are not "well-defined", such as non-measurable functions, transfinite cardinals and ordinals, measurable sets which are not Borel sets, curves in non-metrical analysis situs, is large and is increasing. It is not possible for any one group to guide or stem the tide of mathematical investigation or to say that here is mathematics; thus far and no farther may it go.

In the preliminary remarks to the third chapter Borel points out that in biology the greater interest and value lies in the study of the existent and normal species rather than in the study of hypothetical non-existent or abnormal species. By analogy, in mathematics the subject of functions which increase exponentially (to which that chapter is devoted) is relatively of great importance. The chapter contains ten notes, of which three are on the theory of the increase of integral functions and three are on various modes of approximation. One of the remaining four notes is concerned with an analogy between the number e and the algebraic numbers, one with an analytic partial

differential equation

$$\frac{\partial^2 \varphi}{\partial x^2} - a^4 \frac{\partial^2 \varphi}{\partial y^2} = \psi(x, y)$$

which has for certain values of a a solution $\varphi(x, y)$ which is nowhere analytic in x or y; another to a formula for any analytic integral of a linear partial differential equation with analytic coefficients, and the last to the periods of abelian integrals and a related generalization.

The fourth chapter contains seven notes on the theory of functions of a complex variable which appeared in the COMPTES RENDUS from 1897—1912. If $\varphi(z)$ is a function with zeros at $a_1, a_2, \ldots$, the formula

$$f(z) = \sum \frac{c_n \varphi(z)}{(z - a_n)\varphi'(a_n)}$$

determines, in the case of convergence, a function $f(z)$ which assumes prescribed values c_n for $z = a_n$. In the first note Borel considers conditions that φ and so f be unique and brings out the intimate relationship between the theory of the zeros of integral functions and the problem of determining a function $f(x)$ uniquely by an enumerable infinity of conditions. The next three notes consider the relations between the coefficients of a power series and the singularities of the corresponding function. The fifth note, entitled *Sur l'étude asymptotique des fonctions méromorphes* (1902), is of historical interest. In this note Borel applied the method of exclusion of singularities to the case of functions whose singularities are everywhere dense in a region and was thereby led by a natural sequence of ideas to the discovery of the importance of the sets of measure null. The sixth and seventh notes are concerned respectively with a remarkable and unexplained relation between the coefficients of certain differential equations and the invariants of binary forms and with the variation of an analytic function in the neighborhood of an essential singularity.

In a brief conclusion Borel recommends that mathematicians seek to classify systematically those mathematical entities which are well-defined. He shows clearly that this work will rest upon a profound study and classification of those incommensurable numbers which are truly defineable, and remarks that although the subject is very difficult it is one in which the least conquest is infinitely precious because of its reactions; one attains here the living substance itself of which all mathematical entities are made; nothing is more important than the properties of number.

E. W. CHITTENDEN

SHORTER NOTICES

Vorlesungen über die Theorie der Algebraischen Zahlen. By E. Hecke. Leipzig, Akademische Verlagsgesellschaft, 1923. 264 pp.

This book had its origin in lectures delivered by the author in Basel, Göttingen, and Hamburg. As stated in the preface the aim of the book is to bring the reader to a comprehension of the questions which at present form the summit of the theory of algebraic number fields, without presupposing any knowledge of the theory of numbers. In reading the book the reviewer was particularly impressed by its richness in content. By the skillful coordination of important notions, the author gives, in the 264 pages, an elegant and comprehensive account of the modern theory of algebraic numbers.

The first chapter contains an introductory account of the theory of rational numbers. This is followed by a chapter of 28 pages on the theory of groups with special consideration of abelian groups, both finite and infinite. The purpose of this chapter is to familiarize the reader with those properties of abelian groups which may be advantageously used in the further development of the theory of rational numbers as well as the theory of algebraic numbers.

In the third chapter the group properties developed in the second chapter are used in the further study of the rational numbers. The chapter contains the theory of solutions of congruences; the theory of power residues; and in particular that of quadratic residues leading up to a statement of the law of quadratic reciprocity. The proof of this law is, however, deferred to Chapter eight, which contains the author's proof of the general law of quadratic reciprocity in a general algebraic number field.

Chapter four contains the algebraic development of the theory of algebraic numbers, and the algebraic properties of fields defined by the roots of algebraic equations.

Chapter five is the longest chapter of the book and contains a very comprehensive account of the arithmetic of an algebraic number field. The general theory of ideals is developed and theorems regarding the factorization of rational primes are given for certain special types of fields. This is followed by Minkowski's theorem on linear forms, with its application to the study of the units of a field.

A brief discussion of rings is given, and by the use of fractional ideals the author develops that part of the theory which deals with differents and discriminants of fields, and conductors of rings.

The sixth chapter is an introduction to the use of transcendental methods in the theory of algebraic number fields. It contains the

development of the expression for the density of the prime ideals of a class, and its application to the problem of the determination of the number of classes. This is followed by a consideration of the zeta function and some of its applications.

Chapter seven is devoted to a more detailed consideration of quadratic fields. Norm residues are considered and the properties of Hilbert's norm residue symbol are developed. The separation of the ideals into genera is studied and the number of genera determined.

The further topics considered in this chapter are the zeta function in a quadratic field; the expression for the class number in a quadratic field (this without the use of the zeta function); the Gaussian sum; and the relations existing between ideals of a quadratic number field and binary quadratic forms.

Chapter eight contains a new proof of the most general law of quadratic reciprocity in an arbitrary algebraic number field. In this development the author makes use of his generalization of the Gaussian sum.

The chapter ends with a proof of the existence of class fields of relative degree two. This is a consequence of the general law of reciprocity.

G. E. WAHLIN

Die Philosophischen Grundlagen der Wahrscheinlichkeitsrechnung. By E. Czuber. No. XXIV of Wissenschaft und Hypothese. Leipzig and Berlin, B. G. Teubner, 1923. 8 + 343 pp.

Czuber's book on the philosophy of probability is in essence a clarification of the fundamental concepts and theorems of the theory of probability by means of analysis and criticism of attacks directed against the theory in certain recent writings. The author's patient analysis and kindly criticism center chiefly on Meinong's rather labored *Über Möglichkeit und Wahrscheinlichkeit* and Marbe's naive ideas on probability expressed in his *Die Gleichförmigkeit in der Welt.* The concept and definition of probability, the addition and multiplication theorems, the theorems of Bernoulli, of Poisson, and of Bayes, the relation of probability to our knowledge of nature, all come in for detailed discussion.

Czuber's book, in the opinion of the reviewer, is intended primarily for non-mathematicians. Students of mathematical probability will find in the author's admirable *Wahrscheinlichkeitsrechnung* (3d edition, 1914) the substance of nearly all his views on the fundamentals of probability found in the *Grundlagen.* One instance of departure from his earlier work may be worth noting here: the obtaining (in Chap. VI) of the theorem of Bayes as a simple corollary of the multiplication theorem for dependent events (compare Boole's *Laws of Thought,* p. 254, III).

B. A. BERNSTEIN

Die Arithmetik der Quadratischen Formen, zweite Abteilung. By P. Bachmann. Leipzig, B. G. Teubner, 1923. xxii + 537 pp.

Bachmann's series of books, which together aim to cover the entire field of the theory of numbers, are recognized as the standard from the standpoints both of scholarship and clearness. Hence the publication of a new book in this series is an important event to the many devotees of this field. Although the new volume forms the second part of the work bearing the same title, it is really independent of the first part (which appeared in 1898), as well as of the other volumes of the series. This self-contained book employs largely the methods of the geometry of numbers initiated by Minkowski, and treats mainly the problem of the reduction of quadratic forms.

There are sixteen chapters, the number of pages in each of which varies only slightly from the average 33. The topics of the chapters are as follows: Binary quadratic forms; Lattices and continued fractions; Reduction of indefinite binary quadratic forms; Their minima; Geometrical interpretation of binary quadratic forms by means of lattices in a plane; Lattices in space and positive ternary quadratic forms; Geometry of numbers in n-dimensional space; Positive quadratic forms in n unknowns; Their reduction; Complete forms of Voronoï, limit (or extreme) forms; Equivalence and classes of positive quadratic forms; Decomposable forms; Quadratic and cubic irrationalities; Generalized algorithm of continued fractions; Minkowski's criterion for algebraic numbers; Indefinite quadratic forms in more than two unknowns.

The book was edited by Haussner, but follows essentially the manuscript left by Bachmann at his death in 1920.

L. E. Dickson

Analytische Geometrie des Raumes. By Salmon-Fiedler, revised by Brill-Kommerell. Leipzig, Teubner, 1922. 590 pp.

Those familiar with the original Salmon-Fiedler will at once recognize that about ninety per cent of the revised edition is identical with the original. The remaining ten per cent of the book has been rewritten in such a manner as to make the revision a real contribution as a text book on the subject. The changes have been made in the form of more detailed explanations of certain rather obscure articles, the addition of many cuts and the addition of new subject matter. A great deal of the projective treatment has been omitted and analytic proofs substituted. The subject of quadratic forms has been entirely rewritten and precedes the transformations of surfaces. By changing the order the latter subject is introduced by means of the former. Many new examples have been added and several of the original exercises have been omitted. As a text I consider it superior to the revised English edition.

F. M. Morgan

On a Special Polyadic of Order n—p which can be Derived from any p Independent Vectors in an n-Dimensional Space and which can be Regarded as a Generalization of the Vector Product. By Almar Naess. (Videnskapsselskapets Skrifter, I. Mat.-Naturw. Klasse, 1922, No. 13.) Kristiania, Jacob Dybwad, 1923. 53 pp.

This beautifully printed pamphlet of 53 pages is an important contribution to generalized vector algebra. It deals with the "space-complement" of a set of p vectors, proved invariant under orthogonal transformation, and distributive, i. e. the space-complement of a sum of polyadics is the sum of the space-complements of the terms. The *space-complement of the space-complement* of p vectors is their alternant product, a fact whose importance in the logic of the subject can hardly be overestimated, — e. g. in 3-space the space-complement of two vectors is their vector product, while their alternant product is a skew-symmetric tensor often confounded with the vector product. Many other illustrations are given of which the generalized divergence and curl are especially noteworthy.

F. L. Hitchcock

The Atom and the Bohr Theory of its Structure. By H. A. Kramers and Helge Holst. New York, Knopf, 1923. xiii + 210 pages.

This volume contains an excellent English translation by R. B. Lindsay and Rachel T. Lindsay of the Danish original by the outhors named. There is an introduction by Sir Ernest Rutherford. The latter recommends the book as a clearly written and accurate account of the development of our ideas on atomic structure. My purpose in writing a note about the book in this place is to call the attention of mathematicians to the fact that we have here a self-contained introduction to the important ideas of Bohr and his followers concerning the atomic structure of matter. This book itself is free of mathematical detail. But the theories to which it gives an introduction involve problems of great difficulty and importance requiring the combined skill of the mathematician and the physicist for their resolution. After a brief account of the earlier development of the atomic theory the authors set forth the ideas of Bohr in some detail for the case of the hydrogen atom. The nature of the problems which arise become manifest here in their simplest form. Then several applications are made to various problems of atomic structure. The book will serve to orient the reader in the whole field. Details are not given of the mathematical problems; but one acquainted with as much of the theory as this book contains will find it comfortable to proceed to the memoirs. The Bohr theory opens up a great opportunity for the cooperation of mathematics and physics in solving one of the most fundamental problems of all natural science.

R. D. Carmichael

Vector Analysis. By C. Runge. Vol. I. Translated by H. Levy. New York, Dutton and Co., 1923. vi + 226 pages.

This translation contains the vectors of three-dimensional space. The original was reviewed in this BULLETIN, vol. 27, ser. 2, p. 464. There is little to add with regard to the translation. Heavy black letters have been substituted for German letters, which is more in conformity with prevailing customs, and tends to legibility, and the pages have been printed in a more open manner. The book is printed on thick paper which is a disadvantage as it makes it somewhat clumsy to use, and is not likely to be durable. The reviewer wonders why the translator saw fit (p. 211) to use cogradience and contragradience instead of the common terms cogrediency and contragrediency. The book will be of little use to students of physics particularly.

J. B. SHAW

Champ de Gravitation d'une Sphère Matérielle et Signification Physique de la Formule de Schwarzschild. By Jean Becquerel. Paris, J. Hermann, 1923. 32 pp.

La Théorie Einsteinienne de la Gravitation. By Gustave Mie (translated by J. Rossignol). Paris, J. Hermann, 1922. xi + 118pp.

The first of these publications contains an interesting interpretation of Schwarzschild's equation by means of an orthogonal projection of the non-euclidean solar field upon an asymptotic Minkowskian plane four-space tangent to the solar field at infinity.

This problem is also treated in an appendix which lends distinction to an otherwise undistinguished pamphlet by Mie.

C. N. REYNOLDS, JR.

Annuaire pour l'An 1923. Publié par le Bureau des Longitudes. Paris, Gauthier-Villars, 1923. 654 pp. + Appendices.

Only minor changes are noticeable in the Annuaire for 1923, although these are all of a nature to keep the publication up to date. Since the volume no longer appears well in advance of the year of date, astronomical information for the succeeding year is given in a supplement.

Besides two "Notices", one by P. Appell on Gabriel Lippmann and the other by A. Jobin on Jules Carpentier, a long one entitled *Le climat de France: l'eau atmosphérique* is contributed by G. Bigourdan. This gives in readable form the main features of what we call weather and climate with the physical principles which are to be used in the discussion of them. Particular attention is paid to rainfall, its distribution in time and space, with tables for various departments of France.

E. W. BROWN

NOTES

The concluding number of volume 24 of the TRANSACTIONS OF THIS SOCIETY (December, 1922) contains the following papers: *Representations of a complex point by pairs of ordered real points*, by W. C. Graustein; *Non-loxodromic substitutions and groups in n dimensions*, by E. B. Van Vleck; *Birational transformations simplifying singularities of algebraic curves*, by G. A. Bliss; *A symbolic theory of formal modular covariants*, by Olive C. Hazlett; *On the mean-value theorem corresponding to a given linear homogeneous differential equation*, by Georg Pólya. The third number of volume 25 of the TRANSACTIONS (July, 1923) contains: *Discontinuous boundary conditions and the Dirichlet problem*, by Norbert Wiener; *A type of differential system containing a parameter*, by F. H. Murray; *On a remarkable class of entire functions*, by J. I. Hutchinson; *Note on an ambiguous case of approximation*, by Dunham Jackson; *Expansions in terms of solutions of partial differential equations. Second paper: Multiple Birkhoff series*, by C. C. Camp; *Circular plates of constant or variable thickness*, by C. A. Garabedian; *Permutable rational functions*, by J. F. Ritt; *On approximation by functions of given continuity*, by Dunham Jackson.

The concluding number of volume 45 of the AMERICAN JOURNAL OF MATHEMATICS (October, 1923) contains: *Sylow subgroups in the group of isomorphisms of prime power abelian groups*, by H. A. Bender; *The isodyadic quintic*, by J. C. Glashan; *On class number relations for bilinear forms in four variables*, by E. T. Bell; *Relative inclusiveness of certain definitions of summability*, by D. S. Morse; *Modular invariants of a binary group with composite modulus*, by Constance R. Ballantine; *Determination of a surface by its curvatures and spherical representation*, by W. C. Graustein. The first number of volume 46 (January, 1924) contains: *On the theory of numbers and generalized quaternions*, by L. E. Dickson; *A study of the rational involutorial transformations in space which leave a web of sextic surfaces invariant*, by J. O. Osborn; *On the reduction of differential parameters in terms of finite sets, with remarks concerning differential invariants of analytic transformations*, by O. E. Glenn; *On the isodyadic septimic equations*, by J. C. Glashan.

The first International Congress for Mechanics was held at Delft, Holland, April 22–25, 1924. The congress met in three sections, for the presentation of papers on rational mechanics, theory of elasticity and rigidity, and hydro- and aerodynamics, including aviation. General sessions were also held.

In connection with the International Mathematical Congress to be held in Toronto on August 11-16, the American Section of the International Mathematical Union has extended its membership as provided under the general regulations governing that body. The previous members were as follows:

The President of this Society, ex officio: O. Veblen.

The Secretary of this Society, ex officio: R. G. D. Richardson.

Other representatives of this Society: L. P. Eisenhart, E. V. Huntington, H. F. Blichfeldt.

Representative of the Mathematical Association of America: H. L. Rietz.

The additional members of the Section recently appointed to serve from March to November, 1924, are as follows:

By this Society: A. B. Coble, Arnold Dresden, L. E. Dickson, Virgil Snyder.

By the Mathematical Association of America: R. C. Archibald, E. H. Moore, J. W. Young.

By the National Academy of Sciences: E. H. Moore.

By the American Astronomical Society: E. W. Brown.

By the American Physical Society: Leigh Page.

By the American Association for the Advancement of Science: G. A. Miller.

The Section thus enlarged has elected the following persons from among its members to act as delegates to represent the United States at the quadrennial meeting of the International Mathematical Union, to be held at Toronto during the Congress: Coble, Eisenhart, Huntington, Richardson, Rietz, Snyder. The present officers of the American Section are E. V. Huntington, Chairman; H. L. Rietz, Secretary.

The Royal Society of London will celebrate, on July 10 and 11, the centenary of Lord Kelvin's birth. Sir J. J. Thomson will deliver a memorial address.

Steps are being taken to found a History of Science Society, for encouragement of the study of this subject and for the support of the journal Isis, now edited by Dr. George Sarton. Correspondence with regard to this Society may, for the present, be addressed to Professor David Eugene Smith, 525 West 120th Street, New York City.

The gold medal of the Royal Astronomical Society has been awarded to Professor A. S. Eddington, for his work on star-streaming, on the internal constitution of a star, and on generalized relativity.

Dr. Elihu Thomson, of the General Electric Company, has received the Kelvin gold medal, which was founded in 1914 by British and American engineers and is awarded triennially by the presidents of representative British societies.

Professors Niels Bohr and Albert Einstein have been elected foreign members of the Göttingen Academy of Sciences.

Announcement has been made in this BULLETIN (p. 90, Jan.–Feb., 1924) that the fellowships administered by the National Research Council for physics and chemistry had been extended to include mathematics, and that the extension to mathematics had been placed in effect at once by a supplementary action. Under this action, the following persons have been appointed to National Research Fellowships in mathematics for the academic year 1924–1925 by the Research Fellowship Board: L. M. Graves (Ph. D., Chicago, 1924), Harry Levy, (Ph. D., Princeton, 1924), J. H. Taylor (Ph. D., Chicago, 1924), J. M. Thomas (Ph. D., Pennsylvania, 1923). The following persons have been reappointed as Fellows in mathematical physics: Herman Zanstra (Ph. D., Minnesota, 1923), T. Y. Thomas (Ph. D., Princeton, 1923).

At the request of the Commission on New Types of Examination of the College Entrance Examination Board, Professor L. P. Eisenhart of Princeton University has formed a committee to report on the validity of certain statistical methods used in the investigations of the Commission. The other members of the committee are Professors R. W. Burgess, W. L. Crum, E. V. Huntington, H. H. Mitchell, H. L. Rietz, and J. H. M. Wedderburn.

Mr. W. E. Byrne, of Rennselaer Polytechnic Institute, has been awarded an American Field Service Fellowship for 1924–1925, for the study of mathematics at the University of Paris. Applications for such fellowships for 1925–1926 should reach Dr. I. L. Kandel, 525 West 120th Street, New York City, not later than December 15, 1924.

Dr. Philip Franklin, now Benjamin Peirce teaching fellow at Harvard University, has been appointed instructor in mathematics at the Massachusetts Institute of Technology for the year 1924–1925.

At the University of Chicago, Associate Professor A. C. Lunn has been promoted to a full professorship of mathematics. Dr. Mayme I. Logsdon, instructor in mathematics, has recently been appointed to a deanship in the College of Science.

Dr. Pauline Sperry, instructor in mathematics at the University of California, has been promoted to an assistant professorship.

The death is announced, at the age of ninety-one years, of Gustave Eiffel, the French engineer, known for his work in aerodynamics.

Professor R. E. Wilson, of Northwestern University, died December 30, 1923, at the age of fifty-two. Professor Wilson had been a member of the American Mathematical Society since 1903.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

APPELL (P.). Education et enseignement. Notices et discours. Paris, Alcan, 1922. 2 + 299 pp.

ATANASSIÉVITCH (X.). La doctrine métaphysique et géométrique de Bruno, exposé dans son ouvrage "De triplici minimo." Paris, Les Presses Universitaires de France, 1923. 156 pp.

BOHR (H.) og MOLLERUP (J.). Lærebog i matematisk Analyse. 4: Grundtrek af Læren om de kompleske Tal. København, Gjellerup, 1923. 130 pp.

BOREL (E.). Eléments de la théorie des probabilités. 3e édition, revue et augmentée. Paris, Hermann, 1924. 6 + 246 pp.

COLLINGWOOD (E. F.). See VALIRON (G.).

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